An Introduction to the Schubert Varieties and Toric Varieties

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1 Algebraic Varieties

Let k be an algebraically closed field. $\mathbb{A}^n$ denotes the affine n-space, which is the set $\{(a_1, \ldots a_n) \,|a_i \in k\}$. Let I be an ideal in $k\,[x_1, \ldots x_n]$.

Definition 1 *The set* $V\,(I) = \{(a_1, \ldots a_n) \in \mathbb{A}^n | f\,(a_1, \ldots a_n) = 0 \;\forall\; f \in I\}$ *is an* affine variety*.*

Conversely, let $X \subset \mathbb{A}^n$, define $\mathcal{I}\,(X) = \{f \in k\,[x_1, \ldots x_n]\,| f\,(x) = 0 \;\forall\; x \in X\}$.

Let $\mathbb{P}^n = \{\mathbb{A}^{n+1} \setminus \{0\}\}/\sim$, where $(a_0, \ldots a_n) \sim (\lambda a_0, \ldots \lambda a_n)$ for any $\lambda \in k \setminus \{0\}$. This is the Projective n-space. Now let I be a homogeneous ideal in $k\,[x_0, \ldots x_n]$, meaning I is generated by homogeneous polynomials. In general, it does not make sense to evaluate a polynomial f at a point p of the Projective n-space. However, if f is homogeneous of degree d, then $f\,(\lambda a_0, \ldots \lambda a_n) = \lambda^d f\,(a_0, \ldots a_n)$, and since $\lambda \neq 0$, we can talk about f being zero or non-zero at a point p.

Definition 2 *The set* $V\,(I) = \{(p) \in \mathbb{P}^n | f\,(p) = 0$ *for all homogeneous* $f \in I\}$ *is a* projective variety*.*

Conversely, for $X \subset \mathbb{P}^n$, let $\mathcal{I}\,(X)$ be the ideal generated by $\{f \in k\,[x_0, \ldots x_n]\,,\, f$ homogeneous $|f\,(p) = 0 \;\forall\; p \in X\}$.

Definition 3 *For* X *a projective variety in* $\mathbb{P}^n$*, the* coordinate ring *of* X *is* $k\,[x_1, \ldots x_n]\,/\mathcal{I}\,(X)$*, also denoted* $k\,[X]$*.*

Definition 4 *For X a variety, $f \in k[X]$, the* principal open subsets *are given by $X_f = \{x \in X | f(x) \neq 0\}$.*

Definition 5 *The* Zariski Topology *on $\mathbb{A}^n$ or $\mathbb{P}^n$ has the principal open subsets as a base.*

2 The Grassmannian Variety and its Schubert Varieties

Fix integers $1 \leq d < n$, let $V = k^n$, with basis $\{e_1, \ldots e_n\}$.

Definition 6 *The* Grassmannian *$G_{d,n}$ is the set of d-dimensional subspaces $U \subset V$.*

If $U \in G_{d,n}$ with $a_1, \ldots a_d$ as a basis, then U may be represented by the $n \times d$ matrix $A = (a_{ij})$, of rank d, whose columns are vectors $a_1, \ldots a_d$. This matrix is not unique, as it depends on choice of basis for U.

Definition 7 *Let $I_{d,n} = \{\underline{i} = (i_1, \ldots i_d) | 1 \leq i_1 < \ldots < i_d \leq n\}$.*

Define a partial order $\leq$ on $I_{d,n}$ by defining $\underline{i} \leq \underline{j} \Leftrightarrow i_t \leq j_t \; \forall \, t$. Let $N = \binom{n}{d}$, the order of $I_{d,n}$.

The exterior product map $\Lambda^d : V \oplus \ldots \oplus V \rightarrow V \wedge \ldots \wedge V$ such that $(a_1, \ldots a_d) \mapsto a_1 \wedge \ldots \wedge a_d$ induces an embedding

$$f_d : G_{d,n} \longrightarrow \mathbb{P}\left(\Lambda^d V\right) = \mathbb{P}^{N-1}$$

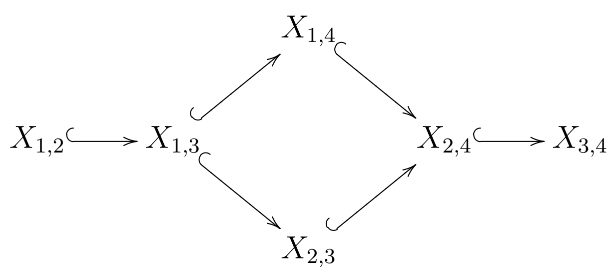
called the Plücker map. We will index the coordinates of $\mathbb{P}^{N-1}$ by the set $I_{d,n}$. Thus, for $p \in \mathbb{P}^{N-1}$, $p_{\underline{i}}$ will denote the $\underline{i}^{th}$ coordinate of p, called the Plücker coordinates. For $U \in G_{d,n}$, represented by matrix A as above, $p_{\underline{i}}(U) = \det(A_{\underline{i}})$, where $A_{\underline{i}}$ represents the $d \times d$ sub-matrix of A with rows $i_1, \ldots i_d$. One can see that this embedding is well-defined, as different representative matrices for the same point will be mapped to the same projective coordinates. Furthermore, the Plücker embedding is injective, so we will often identify $G_{d,n}$ with its embedding in $\mathbb{P}^{N-1}$.

The Plücker embedding identifies $G_{d,n}$ as the zero set of the Plücker quadratic relations. For example, the Plücker quadratic relation for $G_{2,4}$ is $p_{1,4}p_{2,3} = p_{2,4}p_{1,3} - p_{3,4}p_{1,2}$. (There is a general form for the Plücker relations for any Grassmannian which we will leave out here.) We can now assert that the Grassmannian is in fact a Projective variety.

Definition 8 *For* $1 \leq t \leq n$*, let* $V_t \subseteq V$ *spanned by* $\{e_1, \ldots e_t\}$*. For every* $\underline{i} \in I_{d,n}$*, define the Schubert Variety associated to* $\underline{i}$ *as* $X_{\underline{i}} = \{U \in G_{d,n} | \dim(U \cap V_{i_t}) \geq t,\ 1 \leq t \leq d\}$*.*

Remark 1 $X_{\underline{i}} \subseteq X_{\underline{j}} \Leftrightarrow \underline{i} \leq \underline{j}$*.*

<u>Example:</u> $G_{2,4}$



Remark 2 $p_{\underline{j}}|_{X_{\underline{i}}} \neq 0 \Leftrightarrow \underline{i} \geq \underline{j}$*.*

The above remark assures us that the Schubert variety does in fact fit the definition of a projective variety. It's defining ideal is generated by the Plücker quadratic relations and $\{p_{\underline{j}},\ \underline{j} \not\leq \underline{i}\}$.

3 Toric Varieties

To begin, we need to define a *lattice* $N \cong \mathbb{Z}^d$; and a *dual lattice* $M = \mathrm{Hom}\left(N, \mathbb{Z}^d\right) \cong \mathbb{Z}^d$.

Definition 9 *We call* σ *a rational convex polyhedral cone (CPC) generated by* $v_1, \ldots v_n \in N$ *if* $\sigma = \{r_1 v_1 + \ldots + r_n v_n \,|\, r_i \in \mathbb{R}_{\geq 0}\}$*.*

We say $\sigma = C\langle v_1, \ldots v_n \rangle \subset N_{\mathbb{R}}$. Now define $\check{\sigma} = \{u \in M_{\mathbb{R}} |\ \langle u, v \rangle \geq 0,\ \forall v \in \sigma\}$. Let $S_\sigma = \check{\sigma} \cap M$.

Definition 10 *For a rational CPC* σ*, we define an affine variety* $U_\sigma = \mathit{Spec}\,\mathbb{C}[S_\sigma]$*.*

A short definition of Spec: For a ring A, Spec$A = \{$all prime ideal of $A\}$. The Zariski topology is defined by the closed set for any ideal I, $V(I) = \{P \in$Spec$A$ such that $P \supset I\}$. (Defining a variety by a zero set is equivalent to taking the Spec of the coordinate ring.)

Although it might seem unnecessary to define the toric variety using the dual of the CPC, notice that if τ is defined by a subset of the generators of σ, (i.e. $\tau \prec \sigma$), then $S_\sigma \subset S_\tau$, and thus U_τ has a natural embedding in U_σ because Spec is contravariant.

It's also a good place to discuss why these are called toric varieties. Letting T be the algebraic torus $T = (\mathbb{C}^*)^d$, and identifying M with the character group of T, we have a natural action of T on U_σ which is just an extension of the multiplication of T on itself, because T is also embedded in U_σ.

Examples: (My examples are in 2 dimensions, although some can be generalized to higher dimensions.) Let e_1, e_2 be the standard basis in $\mathbb{R}^2$. Let $\sigma = C\langle e_1, e_2\rangle$, and so $\check{\sigma} = \langle e_1^*, e_2^*\rangle$. Then $U_\sigma = \text{Spec}\mathbb{C}[x_1, x_2] = \mathbb{C}^2$.

Let $\tau = C\langle e_1, 2e_2 + e_1\rangle$, then $\tau = C\langle e_2^*, 2e_1^* - e_2^*\rangle$. S_τ will require a third generator, $S_\tau = \langle e_1^*, e_2^*, 2e_1^* - e_2^*\rangle$, so we can write $\mathbb{C}[S_\tau] = \mathbb{C}[x, y, x^2y^{-1}] \cong \mathbb{C}[V, W, Z]/\langle V^2 - WZ\rangle$. U_σ is a "quadric cone."

Definition 11 *A* Fan *is a set of rational CPC's following the rules of a simplicial complex; i.e. all faces of cones are cones in the fan, and any intersection of two cones is a face of each.*

Definition 12 *For a fan Δ, we construct a* toric variety $X(\Delta)$ *by taking the disjoint union of the affine varieties U_σ; and for cones σ and τ in Δ, glue U_σ and U_τ along $U_{\sigma\cap\tau}$.*

Example: $\mathbb{CP}^2$ is a toric variety. Take $v_0 = e_1$, $v_1 = e_2$, $v_2 = -e_1 - e_2$, and let Δ be made up of proper subsets of $\{v_0, v_1, v_2\}$; say $\sigma_0 = C\langle v_0, v_1\rangle$, $\sigma_1 = C\langle v_1, v_2\rangle$, $\sigma_2 = C\langle v_0, v_2\rangle$. Then $U_{\sigma_i} = \mathbb{C}^2$ for each i. We can see that $X(\Delta) = \mathbb{CP}^2$, by associating U_{σ_0} with the subset $\{(x : y : 1) \in \mathbb{CP}^2\}$, $U_{\sigma_1} = \{(x : 1 : z) \in \mathbb{CP}^2\}$, and $U_{\sigma_2} = \{(1 : y : z) \in \mathbb{CP}^2\}$. The intersection of two U_σ's is the subset in which two coordinates are non-zero, and all three affine varieties intersect on the subset where all three coordinates are non-zero, isomorphic to $(\mathbb{C}^*)^2$.