Real division algebras, restricted quiver representations, and Euclidean configurations

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Abstract. A real division algebra is a non-zero real vector space A, endowed with an $\mathbb{R}$-bilinear multiplication $A \times A \to A,\ (x,y) \mapsto xy$ such that for each $a \in A \setminus \{0\}$ both linear operators $L_a : A \to A,\ x \mapsto ax$ and $R_a : A \to A,\ x \mapsto xa$ are invertible. A famous theorem of Hopf (1940) and Bott, Milnor, Kervaire(1958) states that every finite dimensional real division algebra has dimension 1, 2, 4, or 8. The problem of classifying all finite dimensional real division algebras up to isomorphism is solved in the dimensions 1 and 2, but only partially solved in the dimensions 4 and 8.

While these partial solutions historically emerged from diverse approaches and techniques, developed by numerous specialists during half a century, they are at present being understood to follow a common pattern that "locally" relates real division algebras in a first step to modules over an associative algebra, and in a second step to configurations in a Euclidean space, in terms of equivalences of categories. Thus unforeseen connections between non-associative algebras, modules over associative algebras, and Euclidean geometry emerge from the attempt to classify real division algebras. In return, these connections actually enable the classification of certain types of real division algebras.

I will explain this common pattern and exemplify it by revisiting some of the known partial classifications of real division algebras under its unifying perspective.