

S09 MTH 371 LINEAR ALGEBRA  
**FINAL EXAM**

Prof. G.Todorov  
April 17, 2009

Name: \_\_\_\_\_

1. Linear independence/dependence

(a) Show that the following vectors are linearly independent:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Show that the following vectors are linearly dependent and give a linear dependence:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(c) Express the vector  $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$  as a linear combination of the vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

2. Consider the linear transformation given by multiplication by the matrix  $A$  as  $A\vec{x}$ , where

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 1 & 2 & 4 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \text{ and the } rref(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) Find a basis for  $ImA$ .

(b) Find a basis for  $KerA$ .

3. Consider the linear transformation given by multiplication by the matrix  $A$  as  $A\vec{x}$ , where  $A$  is a  $6 \times 14$  matrix, with  $rank(A) = 4$

(a) Fill in the dimensions for this linear transformation:  $\mathbb{R}^{\square} \xrightarrow{A} \mathbb{R}^{\square}$ . Also find:

(b)  $dimIm(A)$

(c)  $dimKer(A)$

(d)  $dim(Im(A))^{\perp}$

(e)  $dimKer(A)^{\perp}$

(f)  $dimIm(A^T)$

(g)  $dimKer(A^T)$

4. Let  $V = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ .

(a) Find a matrix  $A$  such that  $\text{Im}(A) = V$ .

(b) Find  $\text{Proj}_V$ , a matrix of the orthogonal projection onto the subspace  $V$ .

(c) Find a matrix  $B$  such that  $AB = \text{Proj}_V$  and  $BA = \text{Id}$ .

(d) Let  $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ . Find  $B\vec{b}$ .

(e) Find the least squares solution  $\vec{x}^*$  to the equation  $A\vec{x} = \vec{b}$ .

5. Find  $x$  such that  $\det \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & x & 4 & 5 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 0$ .

6. Give an example of a  $8 \times 8$  matrix with real entries, which has eigenvalues:  $\lambda_1 = 8$  with algebraic multiplicity 3, geometric multiplicity 2, and  $\lambda_2 = 5$  with algebraic multiplicity 3, geometric multiplicity 1, and has no other real eigenvalues.

7. Using least squares solution, find the best fitting linear model for the points:  
 $(0, 1), (1, 2), (0, -3), (1, -1)$ .

8. Let  $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(a) Use Gram-Schmidt process to find an orthonormal basis for the image of  $A$ .

(b) Find a  $QR$  factorization of the matrix  $A$ , i.e. write  $A = QR$  where  $Q$  is an orthogonal matrix and  $R$  is an upper triangular matrix.

9. Find the singular value decomposition (SVD) for the matrix  $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ .

10. Consider dynamical system where the state vector  $\vec{x}(t+1)$  can be expressed as  $\vec{x}(t+1) = A\vec{x}(t)$ , where the matrix  $A$  has  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  as an eigenvector with eigenvalue  $\lambda_1 = 3$  and  $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  as an eigenvector with eigenvalue  $\lambda_2 = 4$ .

(a) Find the matrix  $A$ .

(b) Find the matrix  $A^t$  (matrix  $A$  to the power  $t$ )

(c) If the initial state vector is  $\vec{x}(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$  find the closed form for  $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ .