S09 MTH 371 Linear Algebra **NEW PRACTICE QUIZ 3, Solution** Name: ____________________

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Please, justify your answers.

1. Consider the linear transformation given by multiplication by the matrix A, as $A\vec{x}$ for each vector $\vec{x} \in \mathbb{R}^n$.

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & -1 \\ 1 & -3 & -4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boxed{\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m} \qquad \boxed{\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^4}$$

(a) n= ________ 3

(b) m= ________ 4

(c) What is the dimension of the Domain of this transformation? ________ 3

(d) What is the dimension of the Codomain of this transformation?________ 4

(e) Find the $rref A$.

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & -1 \\ 1 & -3 & -4 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -2 \\ 1 & -3 & -1 \\ 1 & -3 & -4 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & -3 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

. So, the leading $\boxed{1}$'s are in the 1st and 3rd column.

(f) Find the image of A, ImA.

$$Im(A) = span\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

This is a minimal such set of vectors, i.e $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -4 \\ 1 \end{bmatrix} \right\}$ forms a basis for $Im(A)$.

(g) Find the kernel of A, $KerA$. From the $rref A$ we have:

$\boxed{x_1} - 3x_2 + 0x_3 = 0$

$\boxed{x_3} = 0$

Solve leading variables in terms of non-leading:

$\boxed{x_1} = 3x_2$

$\boxed{x_3} = 0$

Write the solution in vector form:

$$\vec{x} = \begin{bmatrix} \boxed{x_1} \\ x_2 \\ \boxed{x_3} \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \cdot t$$

$$Ker(A) = span\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This is a minimal such set of vectors, i.e $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis for $Ker(A)$.

2. Find the matrix of rotation $\mathbb{R}^2 \to \mathbb{R}^2$ through the angle $\alpha = \pi/6$.

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

3. Find the matrix of the projection onto the $\{\vec{e}_1, \vec{e}_3\}$ plane in $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Find the matrix of the dilation by a factor of 5 in $\mathbb{R}^2$.

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

5. Find the matrix of the reflection through $\vec{e}_2$ line $\mathbb{R}^2 \to \mathbb{R}^2$.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

6. Consider the vector $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

(a) Find the length $|\vec{v}|$ of $\vec{v}$.
$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{1 \cdot 1 + 5 \cdot 5} = \sqrt{26}$

(b) Find the unit vector $|\vec{u}|$ which determines the same line as the vector $\vec{v}$, i.e. $L_{\vec{u}} = L_{\vec{v}}$.
$\vec{u} = (1/|\vec{v}|)\vec{v} = (1/\sqrt{26}) \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{26} \\ 5/\sqrt{26} \end{bmatrix}$

(c) Find the projection of the vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ onto the line $L_{\vec{v}}$.

$Proj_{L_{\vec{v}}}(\vec{x}) = Proj_{L_{\vec{u}}}(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$.

$Proj_{L_{\vec{v}}}(\vec{e}_1) = Proj_{L_{\vec{u}}}(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{26} \\ 5/\sqrt{26} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{26} \\ 5/\sqrt{26} \end{bmatrix} =$

$(1/\sqrt{26}) \begin{bmatrix} 1/\sqrt{26} \\ 5/\sqrt{26} \end{bmatrix} = \begin{bmatrix} 1/26 \\ 5/26 \end{bmatrix}$.

$$\boxed{A(\vec{e}_1) = \begin{bmatrix} 1/26 \\ 5/26 \end{bmatrix}}$$

(d) Find the projection of the vector $\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ onto the line $L_{\vec{v}}$.

$Proj_{L_{\vec{v}}}(\vec{e}_2) = Proj_{L_{\vec{u}}}(\vec{e}_2) = (\vec{e}_2 \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{26} \\ 5/\sqrt{26} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{26} \\ 5/\sqrt{26} \end{bmatrix} =$

$(5/\sqrt{26}) \begin{bmatrix} 1/\sqrt{26} \\ 5/\sqrt{26} \end{bmatrix} = \begin{bmatrix} 5/26 \\ 25/26 \end{bmatrix}$.

$$\boxed{A(\vec{e}_2) = \begin{bmatrix} 5/26 \\ 25/26 \end{bmatrix}}$$

(e) Find the matrix A of the projection onto the line $L_{\vec{v}}$.

$$\boxed{A = \begin{bmatrix} 1/26 & 5/26 \\ 5/26 & 25/26 \end{bmatrix}}$$

(f) (Maybe for extra credit ??? Not sure.) Use the matrix A, from the previous step to find the matrix B of the reflection through the line $L_{\vec{v}}$.
$B = 2A - I = 2 \begin{bmatrix} 1/26 & 5/26 \\ 5/26 & 25/26 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -24/26 & 10/26 \\ 10/26 & 24/26 \end{bmatrix}$