S09 MTH 371 Linear Algebra **NEW PRACTICE QUIZ 4, SOLUTIONS**

Prof. G.Todorov February 15, 2009

Please, justify your answers.

1. Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & -4 \\ -1 & 5 & -7 \end{bmatrix}$.

   Determine whether the column vectors of A are dependent or independent. If they are independent, say why. If they are dependent, exhibit a linear dependence relation among them.

   • $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & -4 \\ -1 & 5 & -7 \end{bmatrix} \to \begin{bmatrix} -1 & 5 & -7 \\ 0 & 3 & -4 \\ 3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 7 \\ 0 & 3 & -4 \\ 3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 7 \\ 0 & 3 & -4 \\ 0 & 15 & -20 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & -5 & 7 \\ 0 & \boxed{1} & -4/3 \\ 0 & 0 & 0 \end{bmatrix}$

   This is NOT rref A, however it is clear that there are only 2 leading $\boxed{1}$*, so we can conclude that the columns of A are linearly dependent!*

   • *In order to write a linear dependence relation, should find KerA, and therefore rrefA.*

   $\cdots \to \begin{bmatrix} \boxed{1} & 0 & 1/3 \\ 0 & \boxed{1} & -4/3 \\ 0 & 0 & 0 \end{bmatrix}$.

   Leading variables: $\boxed{x_1}, \boxed{x_2}$

   Non-leading variables: x_3

   $\boxed{1}x_1 + 0x_2 + 1/3x_3 = 0$

   $\boxed{1}x_2 + -4/3x_3 = 0$

   $\boxed{x_1} = -1/3x_3$

   $\boxed{x_2} = +4/3x_3$

   General solution for KerA: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/3t \\ 4/3t \\ t \end{bmatrix} = \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \end{bmatrix} t.$

   • *A linear dependence:* $(-1/3)\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + (4/3)\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ -4 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}.$

2. For which value(s) of the constant k do the vectors below form a basis of $\mathbb{R}^4$?

   $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ k \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ k \end{bmatrix}.$

   • *These vectors form a basis for* $\mathbb{R}^4$ *if rank rrefA = 4, where* $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}$.

   $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ k & 4 & 3 & k \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2k+5 \end{bmatrix}$

   $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ *form a basis for* $\mathbb{R}^4$ *for all* $k \neq (5/2), k \in \mathbb{R}$.

3. Consider the liner transformation given by the multiplication by matrix A as $A\vec{x}$.

The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 18 & 20 \end{bmatrix}$ has row-reduced echelon form: $rref A = \begin{bmatrix} \boxed{1} & 0 & -1 & -2 & -3 \\ 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Fill in $\mathbb{R}^{\boxed{5}} \xrightarrow{A} \mathbb{R}^{\boxed{4}}$.

(b) Find the image of A, ImA.

- $ImA = span \left\{ \begin{bmatrix} 1 \\ 6 \\ 11 \\ 16 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 12 \\ 17 \end{bmatrix} \right\}$

(c) Find a basis for ImA.

- (A basis for ImA) $= \left\{ \begin{bmatrix} 1 \\ 6 \\ 11 \\ 16 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 12 \\ 17 \end{bmatrix} \right\}$

(d) Find the dimension of ImA.

- $dim ImA = 2$

(e) Find the kernel of A, $KerA$. (!!!! You need to show your work here !!!!!)

$$KerA = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} s + 2t + 3u \\ -2s - 3t - 4u \\ s \\ t \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \text{ for all } s, t, u \in \mathbb{R} \right\}$$

- $KerA = span \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(f) Find a basis for $KerA$.

- (A basis for $KerA$) $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(g) Find the dimension of $KerA$.

- $dim KerA = 3$

4. Consider the 5x4 matrix $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}$. We are told the vector $\begin{bmatrix} -5 \\ 4 \\ -3 \\ 2 \end{bmatrix}$, is in the kernel of A. Write $\vec{v}_4$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

   - $-5\vec{v}_1 + 4\vec{v}_2 - 3\vec{v}_3 + 2\vec{v}_4 = \vec{0}$ and therefore $\vec{v}_4 = (5/2)\vec{v}_1 - (4/2)\vec{v}_2 + (3/2)\vec{v}_3$.

5. Let V be the subspace of $\mathbb{R}^3$ defined by the equation $3x_1 + 5x_2 - x_3 = 0$.

   (a) Express V as the kernel of a matrix A.

   - $A = \begin{bmatrix} 3 & 5 & -1 \end{bmatrix}$ • $V = KerA = span \left\{ \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$ (!!!!!! Show work !!!!!!)

   (b) Express V as the image of a matrix B.

   - $B = \begin{bmatrix} -5/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. • $V = ImB = span \left\{ \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

6. Let V be the subspace of $\mathbb{R}^3$ defined by the equations
   $x_1 + 2x_2 = 0$
   $3x_1 + 5x_2 - x_3 = 0$.

   (a) Express V as the kernel of a matrix A.

   - $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & -1 \end{bmatrix}$ • $V = KerA = span \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ (!!!!!! Show your work !!!!!!)

   (b) Express V as the image of a matrix B.

   - $B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. • $V = ImB = span \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

7. Find the set of all vectors $\vec{x} \in \mathbb{R}^3$ such that $\left(\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \cdot \vec{x} \right) = 0$.

   - $1x_1 + 2x_2 + 5x_3 = 0$
   - $ker \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ (!!! And again, show your work !!!)

8. Find the set of all vectors $\vec{x} \in \mathbb{R}^3$ such that $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \vec{x} \right) = 0$ and $\left(\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \cdot \vec{x} \right) = 0$.

   - $ker \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 5 \end{bmatrix} = span \left\{ \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix} \right\}$ (!!! And again, show your work !!!)

9. Examples

(a) Give an example of a matrix A with $dimKerA = 2$ and $dimImA = 5$.

$$\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1&0&0&0&0&0&-23\\ 0&1&0&0&0&120&30\\ 0&0&1&0&0&8&0\\ 0&0&0&1&0&3&5\\ 0&0&0&0&1&3&4 \end{bmatrix}$$

Or make your own interesting examples. Change the size of the matrix.

(b) Give an example of a square matrix A with $dimKerA = 2$ and $dimImA = 5$.

$$\begin{bmatrix} 1&0&0&0&0&*&*\\ 0&1&0&0&0&*&*\\ 0&0&1&0&0&*&*\\ 0&0&0&1&0&*&*\\ 0&0&0&0&1&*&*\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$$

Make your own interesting examples. Think how you can make many such matrices.

(c) Give an example of a matrix A with $dimKerA = 0$ and $dimImA = 5$.

$$\begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&*&*&*&*\\ 0&1&*&*&*\\ 0&0&1&*&*\\ 0&0&0&1&*\\ 0&0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ *&*&*&*&*\\ *&*&*&*&* \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix},$$

Make your own interesting examples. Think how you can make many such.

(d) Give an example of a matrix A with $dimKerA = 2$ and $dimImA = 0$.

$\begin{bmatrix} 0&0\\ 0&0 \end{bmatrix}, \begin{bmatrix} 0&0\\ 0&0\\ 0&0 \end{bmatrix}, \ldots$ What else can you make?

(e) Give an example of a square matrix A with $KerA = \mathbb{R}^2$.

$$\begin{bmatrix} 0&0\\ 0&0 \end{bmatrix}$$

(f) Give an example of a square matrix A with $ImA = \mathbb{R}^2$.

$\begin{bmatrix} 1&0\\ 0&1 \end{bmatrix}, \begin{bmatrix} 1&2\\ 0&3 \end{bmatrix}, \begin{bmatrix} a&b\\ c&d \end{bmatrix}, ad-bc \neq 0.$ Anything else?

(g) Give an example of a square matrix A with $ImA \subset \mathbb{R}^3$ and $dimImA = 2$.

$$\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\ 0&0&0\\ 0&1&0 \end{bmatrix}, \begin{bmatrix} 1&*&*\\ 0&1&*\\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 1&2&1\\ 3&4&1\\ 4&6&2 \end{bmatrix}, \begin{bmatrix} 1&2&1\\ 1&2&1\\ 1&2&2 \end{bmatrix},$$

(h) Give an example of an invertible matrix A.

$$[1], [-.3], \begin{bmatrix} 1&2\\ 0&3 \end{bmatrix}, \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\ 0&2&0\\ 0&0&-1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\ 0&a&b\\ 0&c&d \end{bmatrix} ad-bc \neq 0,$$

(i) Give an example of a non-invertible matrix A.

$\begin{bmatrix} 1&2 \end{bmatrix}, [0], \begin{bmatrix} 1&2\\ 1&2 \end{bmatrix},$ All examples from (g). Your examples ?

(j) Give an example of a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$. $\begin{bmatrix} 1&2\\ 1&2\\ 1&2 \end{bmatrix},$

10. True-False

T - F If a 4×4 matrix A has $rankA = 3$ then $dimKerA = 3$. $\boxed{F}$

T - F If a vector $\vec{v}$ is in $KerA$, then the vector $5\vec{v}$ is in $KerA$. $\boxed{T}$

T - F Consider a system of 5 equations in 5 variables. let A be the matrix of coefficients. If the system has ∞ many solutions, then $dimImA = 5$ $\boxed{F}$

T - F Consider a system of 5 equations in 5 variables. let A be the matrix of coefficients. If the system has exactly one solution, then $dimKerA = 5$. $\boxed{F}$

T - F If a 3×3 matrix A has $dimImA = 1$ then $dimKerA = 2$. $\boxed{T}$

T - F If a 5×3 matrix A has $dimImA = 1$ then $dimKerA = 2$. $\boxed{T}$

11. **A**lways-**S**ometimes-**N**ever

(A - S - N) If A is 3×3 matrix, then $dimImA \leq 3$. $\boxed{Always}$

(A - S - N) If A is 4×4 matrix, then $dimImA \leq 3$. $\boxed{Sometimes}$

(A - S - N) If A is 5×5 matrix, then $dimImA = 3$. $\boxed{Sometimes}$

(A - S - N) If A is 5×5 matrix, then $dimImA = 6$. $\boxed{Never}$

(A - S - N) If A is 5×2 matrix, then $dimImA = 6$. $\boxed{Never}$

(A - S - N) If A is 5×2 matrix, then $dimImA + dimKerA = 5$. $\boxed{Never}$

(A - S - N) If a system $A\vec{x} = \vec{0}$ has a non-leading variable, then $KerA$ has dimension at least 1. $\boxed{Always}$

(A - S - N) Let $A\vec{x} = \vec{0}$. Then #{leading variables} + #{non-leading variables}=#{columns of A}. $\boxed{Always}$

(A - S - N) If multiplication by a matrix A, as $A\vec{x}$, defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$ then $dimImA = \#\{\text{leading } \boxed{1}\}$ in the $rrefA$. $\boxed{Always}$

(A - S - N) If multiplication by a matrix A, as $A\vec{x}$, defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$ then $dimImA = 3$. $\boxed{Never}$

(A - S - N) If multiplication by a matrix A, as $A\vec{x}$, defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$ then $dimImA = 2$. $\boxed{Sometimes}$

(A - S - N) If multiplication by a matrix A, as $A\vec{x}$, defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^3$ then $dimImA \leq 2$. $\boxed{Always}$