S09 MTH 371 Linear Algebra **NEW PRACTICE QUIZ 6 Sol** Name: ________

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1. Consider the system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

   (a) Sketch the $Im(A)$ and $\vec{b}$.

   (b) Find the least squares solution $\vec{x}^*$.

   Solve: $A^T A\vec{x}^* = A^T\vec{b}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

   $\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\left[\begin{array}{cc|c} 3 & 2 & 2 \\ 2 & 4 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & .5 \\ 0 & 1 & .25 \end{array}\right]$, $\vec{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} .5 \\ .25 \end{bmatrix}$

   (c) Find $A\vec{x}^*$.

   $A\vec{x}^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \end{bmatrix} = \begin{bmatrix} .5 \\ .5 \\ 1 \end{bmatrix}$

   (d) Sketch $A\vec{x}^*$ (above).

   (e) Find $\vec{b} - A\vec{x}^*$.

   $\vec{b} - A\vec{x}^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} .5 \\ .5 \\ 1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \\ 0 \end{bmatrix}$

   (f) Check that $(\vec{b} - A\vec{x}^*) \perp Im(A)$.

   $\left(\begin{bmatrix} .5 \\ -.5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = 0$, and $\left(\begin{bmatrix} .5 \\ -.5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right) = 0$

   (g) Sketch $\vec{b} - A\vec{x}^*$ (above).

   (h) Find $|\vec{b} - A\vec{x}^*|$, the length of $\vec{b} - A\vec{x}^*$.

   $|\vec{b} - A\vec{x}^*| = \sqrt{(.5)^2 + (-.5)^2 + 0^2} = \sqrt{.5}$

   (i) Find the error which occurs by using the least squares solution.

   $\sqrt{.5}$

   (j) Show the error on your sketch.(above)

2. Consider the points: (0,1), (0,0), (2,1).

   (a) Sketch these points.

   (b) Find the best fitting linear model $f(t) = c_0 + c_1 t$ through these points using least squares.

   $(0,1) \implies f(0) = 1 \implies \quad c_0 + c_1 \cdot 0 = 1$

   $(0,0) \implies f(0) = 0 \implies \quad c_0 + c_1 \cdot 0 = 0$ \quad Solve $A\vec{c} = \vec{b}$.

   $(2,1) \implies f(2) = 1 \implies \quad c_0 + c_1 \cdot 2 = 1$

   Consider the system $A\vec{c} = \vec{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

   Find the least squares solution $\vec{c}^*$.

   Solve: $A^T A \vec{c}^* = A^T \vec{b}$.

   Do the work and get: $\vec{c}^* = \begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix} = \begin{bmatrix} .5 \\ .25 \end{bmatrix}$.

   The best fitting line through the above points is: $f(t) = .5 + .25t$.

   (c) Sketch the line that you obtained as the best fitting linear model on the above figure.

3. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}$ and let $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   (a) Find the least squares solution $\vec{x}^*$ to the equation $A\vec{x} = \vec{b}$.

   Solve: $A^T A \vec{x}^* = A^T \vec{b}$.

   The matrix A has $rankA = 2$. Therefore $A^T A$ is invertible.

   $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$.

$$(A^T A)^{-1} A^T = \left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} = (1/11) \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/11 & 6/11 & 5/11 & 2/11 \\ 2/11 & 1/11 & -1/11 & 4/11 \end{bmatrix}$$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 1/11 & 6/11 & 5/11 & 2/11 \\ 2/11 & 1/11 & -1/11 & 4/11 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/11 \\ 4/11 \end{bmatrix}$$

   (b) Find the matrix of orthogonal projection $Proj_{ImA}$ onto the subspace ImA.

$$Proj_{ImA} = A(A^T A)^{-1} A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/11 & 6/11 & 5/11 & 2/11 \\ 2/11 & 1/11 & -1/11 & 4/11 \end{bmatrix} =$$

$$\begin{bmatrix} 2/11 & 1/11 & -1/11 & 4/11 \\ 1/11 & 6/11 & 5/11 & 2/11 \\ -1/11 & 5/11 & 6/11 & -2/11 \\ 4/11 & 2/11 & -2/11 & 8/11 \end{bmatrix} = (1/11) \begin{bmatrix} 2 & 1 & -1 & 4 \\ 1 & 6 & 5 & 2 \\ -1 & 5 & 6 & -2 \\ 4 & 2 & -2 & 8 \end{bmatrix}$$

   (c) Find the matrix B such that $AB = Proj_{ImA}$ and $BA = I_2$, the 2×2 identity matrix.

$$B = (A^T A)^{-1} A^T = \begin{bmatrix} 1/11 & 6/11 & 5/11 & 2/11 \\ 2/11 & 1/11 & -1/11 & 4/11 \end{bmatrix}$$

4. Consider the linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$ given by multiplication by the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 0 & 10 \end{bmatrix}$.

(a) Find $Im(A)$

$A = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 0 & 10 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore

$Im(A) = span \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\} \subset \mathbb{R}^2$.

(b) Sketch $Im(A)$ and $(Im(A))^\perp$

(c) Find $Ker(A^T)$ and sketch on the same figure.

$$Ker(A^T) = Ker\left(\begin{bmatrix} 1 & 5 \\ 0 & 0 \\ 2 & 10 \end{bmatrix}\right), \begin{bmatrix} 1 & 5 \\ 0 & 0 \\ 2 & 10 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} \boxed{1} & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \boxed{x_1} + 5x_2 = 0, \boxed{x_1} = -5x_2.$$

$Ker(A^T) = span \left\{ \begin{bmatrix} -5 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^2$

(d) Find $Im(A^T)$.

$Im(A^T) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^3$

(e) Sketch $Im(A^T)$ and $(Im(A^T))^\perp$

(f) Find $Ker(A)$ and sketch on one of the above figures.

$A = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 0 & 10 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\boxed{x_1}$ leading variable, x_2, x_3 non-leading variables $\boxed{x_1} = -2x_3$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t.$$

$Ker(A) = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3$

5. Let A be a 3×7 matrix with $rankA = 2$. Consider the linear transformation defined by multiplication by A as $A\vec{x}$. Find the following:

$$\mathbb{R}^7 \xrightarrow{A} \mathbb{R}^3 \qquad \mathbb{R}^3 \xrightarrow{A^T} \mathbb{R}^7 \qquad \mathbb{R}^7 \xrightarrow{A^TA} \mathbb{R}^7 \qquad \mathbb{R}^3 \xrightarrow{AA^T} \mathbb{R}^3$$

(a) $dimIm(A) = 2$, $\quad (dimIm(A) = rankA)$

(b) $dimKer(A) = 5$, $\quad (dimIm(A) + dimKer(A) = 7)$

(c) $dim(Im(A))^{\perp} = 1$, $\quad (dimV + dim(V)^{\perp} = 3$, for subspace $V \subset \mathbb{R}^3)$

(d) $dim(Ker(A))^{\perp} = 2$, $\quad (dimV + dim(V)^{\perp} = 7$, for subspace $V \subset \mathbb{R}^7)$

(e) $dimIm(A^T) = 2$, $\quad (dimIm(A^T) = rankA^T = rankA)$

(f) $dimKer(A^T) = 1$, $\quad (Ker(A^T) = (ImA)^{\perp})$ or $(dimIm(A^T) + dimKer(A^T) = 3)$

(g) $dim(Im(A^T))^{\perp} = 5$, $\quad (dimV + dim(V)^{\perp} = 7$, for subspace $V \subset \mathbb{R}^7)$

(h) $dim(Ker(A^T))^{\perp} = 2$, $\quad (dimV + dim(V)^{\perp} = 3$, for subspace $V \subset \mathbb{R}^3)$

(i) $rank(A^TA) = 2$, $\quad (rank(A^TA) = rankA)$

(j) $rank(AA^T) = 2$, $\quad ((A^T)^T = A, rank(AA^T) = rankA^T)$

(k) $dimIm(A^TA) = 2$

(l) $dimKer(A^TA) = 5$

(m) $dim(Im(A^TA))^{\perp} = 5$

(n) $dim(Ker(A^TA))^{\perp} = 2$

(o) $dimIm(AA^T) = 2$

(p) $dimKer(AA^T) = 1$

(q) $dim(Im(AA^T))^{\perp} = 1$

(r) $dim(Ker(AA^T))^{\perp} = 2$