S09 MTH 371 Linear Algebra **NEW PRACTICE QUIZ 7(b)** Name: ________

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1. For each of the following, either give an example (or several if asked), or explain why it does not exist. All matrices on this quiz are with entries in $\mathbb{R}$ and questions are about real eigenvalues (i.e. in $\mathbb{R}$).

   (a) 2×2 matrix with two different eigenvalues.

   (b) 2×2 matrix (which is not diagonal) with two different eigenvalues.

   (c) 2×2 matrix with eigenvalues 4 and 5.

   (d) 2×2 matrix with one eigenvalue of algebraic multiplicity 2.

   (e) 2×2 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 3. $\boxed{\textit{impossible}}$

   (f) 2×2 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 2.

   (g) 2×2 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.

   (h) 2×2 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 0. $\boxed{\textit{impossible}}$

   (i) 2×2 matrix with no real eigenvalues.

   $$\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{array}{c} \textit{such that the characterictic polynomial has no zeros,} \\ \textit{e.g.rotations}(\neq k\pi)\textit{followed by dilations, and many other} \end{array}}$$

   $$\boxed{\begin{array}{c} \textit{Just a few specific examples:} \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -5 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -100 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -5 \\ 200 & 3 \end{bmatrix} \end{array}}$$

   (j) 3×3 matrix with three different eigenvalues.

   (k) 3×3 matrix with exactly two different eigenvalues.

   (l) 3×3 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 3. $\boxed{\textit{impossible}}$

   (m) 3×3 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 2, and o other eigenvalues. $\boxed{\textit{impossible}}$

   (n) 3×3 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1, and no other eigenvalues. $\boxed{\textit{impossible}}$

   (o) 3×3 matrix with one eigenvalue of algebraic multiplicity 2 and geometric multiplicity 0. $\boxed{\textit{impossible}}$

   (p) 3×3 matrix with $\lambda_1 = 5$ an eigenvalue with algebraic and geometric multiplicity 2 and $\lambda_2 = 4$ an eigenvalue with algebraic and geometric multiplicity 1.

   (q) 3×3 matrix with $\lambda_1 = 6$ an eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1 and $\lambda_2 = 7$ an eigenvalue with algebraic and geometric multiplicity 1.

   (r) 3×3 matrix with $\lambda_1 = 8$ an eigenvalue with algebraic and geometric multiplicity 2 and $\lambda_2 = -3$ an eigenvalue with algebraic multiplicity 1 and geometric multiplicity 2. $\boxed{\textit{impossible}}$

(s) 3×3 matrix having exactly one eigenvalue with algebraic multiplicity 1 (and no other eigenvalues). $\boxed{A = \begin{bmatrix} f & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}, \textit{where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \textit{ is any matrix from part } (i)}$

Just a few specific examples:
$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & -100 \\ 0 & 1 & 3 \end{bmatrix}$$

(t) 3×3 matrix with no eigenvalues. *impossible*

(u) 4×4 matrix with no eigenvalues.

$$\boxed{A = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}, \textit{where } \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \textit{ are any matrices like in part } (i)}$$

(v) 5×5 matrix with no eigenvalues. *impossible*

(w) 3×3 matrix with 6 eigenvalues.

2. True-False (circle the correct answer)

$\boxed{T}$ - F Characteristic polynomial of a 5×5 matrix has degree 5.

$\boxed{T}$ - F A 5×5 matrix has at least one eigenvalue.

T - $\boxed{F}$ A 5×5 matrix must have 5 different eigenvalues.

$\boxed{T}$ - F Algebraic multiplicity of any eigenvalue is ≥ 1.

$\boxed{T}$ - F Geometric multiplicity of any eigenvalue is ≥ 1.

$\boxed{T}$ - F For any eigenvalue, the algebraic multiplicity is $\geq$ geometric multiplicity.

T - $\boxed{F}$ If A is a 5×5 matrix, then the sum of algebraic multiplicities must be 5.

$\boxed{T}$ - F If A is a 5×5 matrix, then the sum of algebraic multiplicities might be 5.

T - $\boxed{F}$ If A is a 5×5 matrix, then the sum of geometric multiplicities must be 5.

$\boxed{T}$ - F If A is a 5×5 matrix, then the sum of geometric multiplicities might be 5.

$\boxed{T}$ - F Geometric multiplicity of an eigenvalue is equal to the dimension of the corresponding eigenspace.

$\boxed{T}$ - F For each eigenvalue, there is at least one eigenvector with that eigenvalue.

$\boxed{T}$ - F If A is a 5×5 matrix, then for each eigenvalue there are infinitely many eigenvectors with that eigenvalue.

Any non-zero multiple of an eigenvector is again eigenvector with the same eigenvalue, so you have infinitely many of them, but only finitely many linearly independent!

T - $\boxed{F}$ If A is a 5×5 matrix, then for each eigenvalue, there are always 5 linearly independent eigenvectors with that eigenvalue.

$\boxed{T}$ - F If A is a 5×5 matrix, then for each eigenvalue, there are at most 5 linearly independent eigenvectors with that eigenvalue.

3. Diagonalize the following matrices if possible. If it is impossible, explain why.

(a) $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

$$\boxed{\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}}$$

(b) $\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$

$p_A(\lambda) = \lambda^2 - 5\lambda + 8$ *has no roots since* $(-5)^2 - 4 \cdot 1 \cdot 8 < 0$

$\Longrightarrow$ *no eigenvalues*

$\Longrightarrow$ *no eigenbasis*

$\Longrightarrow$ *not diagonalizable*

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

$p_A(\lambda) = \lambda(\lambda-2)^2$ *has roots:*	$\lambda_1 = 0$	$\lambda_2 = 2$
with algebraic multiplicities:	1	2
with geometric multiplicities:	1	1
sum of geometric multiplicities:	$1 + 1 = 2 < 3$	

$\Longrightarrow$ *no eigenbasis*

$\Longrightarrow$ *not diagonalizable*

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$

$p_A(\lambda) = \lambda^2(\lambda-2)$ *has roots:*	$\lambda_1 = 0$	$\lambda_2 = 2$
with algebraic multiplicities:	2	1
with geometric multiplicities:	2	1
sum of geometric multiplicities:	$2 + 1 = 3$	

$\Longrightarrow$ *there is an eigenbasis*

$\Longrightarrow$ *matrix is diagonalizable*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

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Just a small computational detail that somebody asked:

$$E_2 = Ker(\lambda I - A) = Ker(2I - A) =$$

$$Ker \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 2 \end{bmatrix} = Ker \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 2 \end{bmatrix} =$$

$$Ker \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} = Ker \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = span \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = span \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$