

1. Consider the polynomial $p(x) = x^2 + x + 2 \in \mathbb{Z}_3[x]$. Let $K = \mathbb{Z}_3[x]/(x^2 + x + 2)$ be the congruence-class ring.
 - (a) Prove that K is a field.

 - (b) List all the elements of the field K .

 - (c) How many elements are there in K .

 - (d) Consider the polynomial ring $K[x]$ and consider $p(x)$ as an element of $K[x]$. (Write $p(x)$ with coefficients in K .)

 - (e) Prove that the element $[x] \in K$ is a root of the polynomial $p(x) \in K[x]$.

 - (f) Prove that the element $[2x + 2] \in K$ is a root of the polynomial $p(x) \in K[x]$.

 - (g) Factor the polynomial $p(x) = (x - a)(x - b) \in K[x]$, i.e. with $a, b \in K$.

2. If $p(x)$ is an irreducible quadratic polynomial in $F[x]$, show that $F[x]/(p(x))$ contains all the roots of $p(x)$.

3. Examples of rings and fields:

- (a) Give an example of a finite ring which is not a field.
- (b) Give an example of a finite ring which is a field.
- (c) Give an example of a infinite ring which is not a field.
- (d) Give an example of a infinite ring which is a field.
- (e) Give an example of a ring with 4 elements which is not a field.
- (f) Give an example of a ring with 4 elements which is a field.
- (g) Give an example of a ring with 8 elements which is not a field.
- (h) Give an example of a ring with 8 elements which is a field.
- (i) Give two examples of a ring with 128 elements which is not a field.
- (j) Give an example of a ring with 32 elements which is a field.
- (k) Give an example of a ring with 9 elements which is not a field.
- (l) Give an example of a ring with 9 elements which is a field.
- (m) Give an example of a ring (which is not a field) which contains \mathbb{Z}_5 as a subring.
- (n) Give an example of a field which contains \mathbb{Z}_5 as a subring.

4. Rings R without identity 1_R .

- (a) Show that $3\mathbb{Z} = \{0, \pm 3, \pm 6, \dots\}$ with addition and multiplication from \mathbb{Z} is a ring without identity.

- (b) Show that $n\mathbb{Z} = \{0, \pm n, \pm 2n, \dots\}$ with addition and multiplication from \mathbb{Z} is a ring without identity, for $n \neq \pm 1$.

- (c) Let R be a ring. Show that any ideal $I \subsetneq R$ with addition and multiplication from R is a ring. (This is usually ring without identity.)

- (d) Give an example of a ring R and an ideal $I \subsetneq R$ with addition and multiplication from R which is a ring with identity.

- (e) Let R be a ring. Show that any left ideal $I \subsetneq R$ with addition and multiplication from R is a ring. (This is usually ring without identity.)

- (f) Give an example of a non-commutative ring R and a left ideal $I \subsetneq R$ with addition and multiplication from R which is a ring with identity.

- (g) Let R be a ring. Show that any right ideal $I \subsetneq R$ with addition and multiplication from R is a ring.

- (h) Give an example of a commutative ring and a subring which is not an ideal.

- (i) Give an example of a non-commutative ring and a subring which is neither left nor right ideal.

5. Ring homomorphisms

- (a) Let R be a ring. Then $f : R \rightarrow R$ given by $f(x) = x$ is a ring homomorphism. What is $Im(f)$? What is $Ker(f)$?
- (b) Let R be a ring. Then $f : R \rightarrow R$ given by $f(x) = 0_R$ is a ring homomorphism. What is $Im(f)$? What is $Ker(f)$?
- (c) Let R, R' be a ring. Then $f : R \rightarrow R'$ given by $f(x) = 0_{R'}$ is a ring homomorphism. What is $Im(f)$? What is $Ker(f)$?
- (d) Let R be a ring and I an ideal. Then $\pi : R \rightarrow R/I$ given by $\pi(r) = r + I$ is a ring homomorphism. What is $Im(\pi)$? What is $Ker(\pi)$?
- (e) Consider the map $f : \mathbb{Z} \rightarrow \mathbb{Z}_8$ defined as $f(a) := [a]_8$, the congruence class of a mod 8.
- What is $Im(f)$?
 - What is $Ker(f)$?
 - Prove that $\mathbb{Z}/(8\mathbb{Z}) \cong \mathbb{Z}_8$
- (f) Consider the map $f : \mathbb{Z} \rightarrow \mathbb{Z}_n$ defined as $f(a) := [a]_n$, the congruence class of a mod n .
- What is $Im(f)$?
 - What is $Ker(f)$?
 - Prove that $\mathbb{Z}/(n\mathbb{Z}) \cong \mathbb{Z}_n$
- (g) Consider the map $f : \mathbb{Z}_{24} \rightarrow \mathbb{Z}_8$ defined as $f([n]_{24}) := [n]_8$, the congruence classes of n mod 24 and 8 (respectively).
- Prove that f is well defined.
 - Prove that f is ring homomorphism.
 - What is $Im(f)$?
 - What is $Ker(f)$?
 - Apply the First Isomorphism Theorem - write the isomorphism.
 - Consider the ideals $24\mathbb{Z} \subset 8\mathbb{Z} \subset \mathbb{Z}$. Apply the Third Isomorphism Theorem.

6. More ring homomorphisms

- (a) Let R and S be rings and $R \times S$ product ring. Prove that the map $f : R \rightarrow R \times S$ given by $f(r) = (r, 0_S)$ is a ring homomorphism. What is $Im(f)$? What is $Ker(f)$?
- (b) Prove that the map $f : \mathbb{Z}_8 \rightarrow \mathbb{Z}_8 \times \mathbb{Z}_3$ given by $f([a]_8) = ([a]_8, [0]_3)$ is a ring homomorphism. What is $Im(f)$? What is $Ker(f)$?
- (c) Prove that the map $f : \mathbb{Z}_8 \rightarrow \mathbb{Z}_{24}$ given by $f([a]_8) = [9a]_{24}$ is a ring homomorphism. What is $Im(f)$? What is $Ker(f)$?
- (d) Prove that the map $f : \mathbb{Z} \rightarrow \mathbb{Z}_{24}$ given by $f(a) = [9a]_{24}$ is a ring homomorphism. What is $Im(f)$? What is $Ker(f)$?

7. Some maps which are NOT ring homomorphisms. Prove it.

- (a) $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(a) = 3a$ is not a ring homomorphism. Is it a group homomorphism?
- (b) $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(a) = 3 + a$ is not a ring homomorphism. Is it a group homomorphism?
- (c) $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(a) = na$ is not a ring homomorphism ($n \neq 1$). Is it a group homomorphism?
- (d) $f : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ given by $f(a) = (a, 1)$ is not a ring homomorphism. Is it a group homomorphism?
- (e) $f : \mathbb{Z} \rightarrow \mathbb{Z}_4$ given by $f(a) = [3a]_4$ is not a ring homomorphism. Is it a group homomorphism?

8. Let R be a ring with 1_R . Let $f : R \rightarrow S$ be a ring homomorphism. Then:

- (a) $f(1_R)$ is an idempotent in S . (e is an idempotent if $e^2 = e$)
- (b) $f(1_R)$ is an identity in the ring $Im(f)$

9. Which of the following maps are ring homomorphisms?

(a) $f : \mathbb{Z} \rightarrow \mathbb{Q}$ given by $f(a) = a$

(b) $f : \mathbb{Z} \rightarrow \mathbb{Q}$ given by $f(a) = 2a$

(c) $f : \mathbb{Z} \rightarrow \mathbb{Z}_5$ given by $f(a) = [a]_5$

(d) $f : \mathbb{Z} \rightarrow \mathbb{Z}_5$ given by $f(a) = [2a]_5$

(e) $f : \mathbb{Z} \rightarrow \mathbb{Z}_6$ given by $f(a) = [a]_6$

(f) $f : \mathbb{Z} \rightarrow \mathbb{Z}_6$ given by $f(a) = [2a]_6$

(g) $f : \mathbb{Z} \rightarrow \mathbb{Z}_6$ given by $f(a) = [3a]_6$

(h) $f : \mathbb{Z} \rightarrow \mathbb{Z}_6$ given by $f(a) = [4a]_6$

(i) $f : \mathbb{Z} \rightarrow \mathbb{Z}_6$ given by $f(a) = [5a]_6$

(j) $f : \mathbb{Z}_3 \rightarrow \mathbb{Z}_6$ given by $f([a]_3) = [a]_6$

(k) $f : \mathbb{Z}_3 \rightarrow \mathbb{Z}_6$ given by $f([a]_3) = [2a]_6$

(l) $f : \mathbb{Z}_3 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3$ given by $f([a]_3) = ([0]_2, [a]_3)$

(m) $f : \mathbb{Z}_3 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3$ given by $f([a]_3) = ([1]_2, [a]_3)$

(n) $f : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_3$ given by $f([a]_2) = ([a]_2, [2]_3)$