S09 MTH 576 Rings Fields **QUIZ 6** Name: ____________________

Prof. G.Todorov $March\ 27, 2009$

Some questions are really easy, some are just definitions, some are just restatements of theorems, some are repetitions in different form ...

Consider the polynomial $f(x) = x^4 - 4 \in \mathbb{Q}[x]$. Let $f_1(x) = x^2 - 2 \in \mathbb{Q}[x]$ and $f_2(x) = x^2 + 2 \in \mathbb{Q}[x]$.

1. Prove that $f_1(x) = x^2 - 2 \in \mathbb{Q}[x]$ is irreducible.

2. Find the splitting field K_1 of $f_1(x) = x^2 - 2 \in \mathbb{Q}[x]$ (i.e. over $\mathbb{Q}$).

3. Find a $\mathbb{Q}$ vector space basis for K_1.

4. Find $dim_{\mathbb{Q}} K_1$.

5. Find $[K_1 : \mathbb{Q}]$.

6. Prove that $i\sqrt{2}$ is algebraic over $\mathbb{Q}(\sqrt{2})$.

7. Find the minimal polynomial for $i\sqrt{2}$ over $\mathbb{Q}(\sqrt{2})$.

8. Find a $\mathbb{Q}(\sqrt{2})$-basis for $(\mathbb{Q}(\sqrt{2}))(i\sqrt{2})$.

9. Prove that $f_2(x) = x^2 + 2 \in \mathbb{Q}[x]$ is irreducible.

10. Prove that $i\sqrt{2}$ is algebraic over $\mathbb{Q}$.

11. Find the minimal polynomial for $i\sqrt{2}$ over $\mathbb{Q}$.

12. Find a $\mathbb{Q}$-basis for $\mathbb{Q}(i\sqrt{2})$.

13. Find $[\mathbb{Q}(i\sqrt{2}) : \mathbb{Q}]$.

14. Prove that $\sqrt{2}$ is algebraic over $\mathbb{Q}(i\sqrt{2})$.

15. Find the minimal polynomial for $\sqrt{2}$ over $\mathbb{Q}(i\sqrt{2})$.

16. Find a $\mathbb{Q}(i\sqrt{2})$-basis for $(\mathbb{Q}(i\sqrt{2}))(\sqrt{2})$.

17. Prove that $\left(\mathbb{Q}(i\sqrt{2})\right)(\sqrt{2}) = \mathbb{Q}(i\sqrt{2}, \sqrt{2}) = (\mathbb{Q}(\sqrt{2}))(i\sqrt{2})$

18. Prove that $\mathbb{Q}(i\sqrt{2}, \sqrt{2}) = \mathbb{Q}(\sqrt{2}, i)$

19. Prove that $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$

20. Prove that $\sqrt{2} + i$ is algebraic over $\mathbb{Q}$.

21. Find the minimal polynomial for $i + \sqrt{2}$ over $\mathbb{Q}$.

22. Find a $\mathbb{Q}$-basis for $\mathbb{Q}(i + \sqrt{2})$.

23. Find $[\mathbb{Q}(i + \sqrt{2}) : \mathbb{Q}]$.

24. Consider:

- $\mathbb{Q}(\sqrt{2})$,
- $\mathbb{Q}(i\sqrt{2})$,
- $\mathbb{Q}(\sqrt{2}, i\sqrt{2})$,
- $\mathbb{Q}(\sqrt{2}, i)$,
- $\mathbb{Q}(\sqrt{2} + i)$,
- $\mathbb{Q}[x]/(x^2 - 2)$,
- $\mathbb{Q}[x]/(x^2 + 2)$,
- $\mathbb{Q}(\sqrt{2})[x]/(x^2 + 2)$,
- $\mathbb{Q}(i\sqrt{2})[x]/(x^2 - 2)$,
- $\mathbb{Q}[x]/(x^2 - 2)(x^2 + 2)$,
- ... anything else you can add.

25. Indicate clearly on ONE chart:

- Which fields are isomorphic,
- Which are field extensions,
- Degree of extension,
- One of the above is not a field, which one?