SINGULAR LOCI OF BRUHAT HIBI TORIC VARIETIES

PRESENTATION BY JUSTIN BROWN

In this talk I will discuss the proof to the following conjecture.

Conjecture . *For $\mathcal{L}$ a minuscule lattice,*

$$Sing\, X(\mathcal{L}) = \bigcup_{(\alpha,\beta)} Z_{\alpha,\beta},$$

where (α,β) is a skew pair of join-meet irreducibles in $\mathcal{L}$, and $Z_{\alpha,\beta} = \{P \in X(\mathcal{L}) \mid P_\theta = 0, \forall\, \theta \in I_{\alpha,\beta}\}$, where $I_{\alpha,\beta} = [\alpha \wedge \beta, \alpha \vee \beta]$.

This conjecture began with $\mathcal{L}$ the lattice $I_{d,n}$. In 1996, Lakshmibai and Gonciulea made the conjecture, and proved the $\supset$ inclusion. In 2000, V. Batyrev et. al. proved the $\subset$ inclusion. Our goal was to give a proof for $\mathcal{L}$ any minuscule lattice, using the conversion to convex polyhedral cones.

1. Grid Lattices

Give $\mathbb{N} \times \mathbb{N}$ the lattice structure

$$(\alpha_1,\alpha_2) \wedge (\beta_1,\beta_2) = (\delta_1,\delta_2),\ (\alpha_1,\alpha_2) \vee (\beta_1,\beta_2) = (\gamma_1,\gamma_2),$$

where $\delta_i = \min\{\alpha_i,\beta_i\}$, $\gamma_i = \max\{\alpha_i,\beta_i\}$.

Definition 1.1. Let J be a finite, distributive sublattice of $\mathbb{N}\times\mathbb{N}$, such that if α covers β in J, then α covers β in $\mathbb{N}\times\mathbb{N}$ as well. Then we say J is a *grid lattice.*

For the rest of this section, let J be a grid lattice, and let $\mathcal{L}$ be the poset of ideals of J. (Show transparency)

Let α, β be two non-comparable join-meet irreducibles in $\mathcal{L}$. Thus α, β are meet irreducibles in J, this follows from the fact that the meet of two join irreducibles is a join irreducible (when J is a distributive lattice), and in fact is the same element in both $\mathcal{L}$ and J. Let $\alpha\wedge\beta = \mu$, and thus $\mu \in J$.

For a pair of skew join-meet irreducible elements α, $\beta \in \mathcal{L}$, let $I_{\alpha,\beta} = [\alpha \wedge \beta, \alpha \vee \beta] \subset \mathcal{L}$. Let $\mathcal{L}_{\alpha,\beta} = \mathcal{L} \setminus I_{\alpha,\beta}$.

Date: June 18, 2007.

Proposition 1.2. *$\mathcal{L}_{\alpha,\beta}$ is an embedded sublattice.*

Definition 1.3. A face τ of σ is a *singular* (resp. *non-singular*) face if P_τ is a singular (resp. non-singular) point of X_σ. (Note that P_τ is a singular point of X_σ if X_τ is a singular variety.)

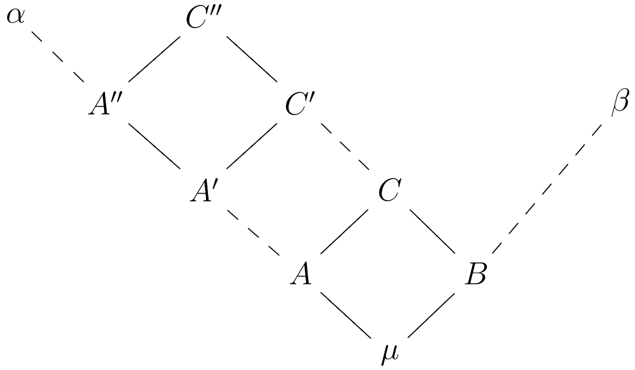
$$W = \{e_z, z \in M(J(\mathcal{L})),\ e_{y'} - e_y, (y, y') \in Z(J(\mathcal{L}))\}.$$

Definition 1.4. Let us denote by W the set of generators for σ as described above. Define

$$W(\tau) = \{v \in W \mid f_{I(\alpha)}(v) = 0, \forall\, \alpha \in D_\tau\}.$$

Then $W(\tau)$ gives a set of generators for any face τ of σ.

Let σ be the cone associated to $X(\mathcal{L})$. Let $\tau_{\alpha,\beta}$ be the face of σ such that $D_{\tau_{\alpha,\beta}} = \mathcal{L}_{\alpha,\beta}$.



Theorem 1.5. *Following the notation from above, $W(\tau) = \{e_\mu - e_A,\ e_\mu - e_B,\ e_A - e_C,\ e_B - e_C\}$.*

Remark 1.6. For $\tau = \langle e_\mu - e_A, e_\mu - e_B, e_A - e_C, e_B - e_C\rangle$ $X_\tau = \mathrm{Spec}K[a, b, c, d]/\langle ad - bd\rangle$.

Thus $X(\mathcal{L}_{\alpha,\beta}) \subset SingX(\mathcal{L})$. In the framework of Hibi toric varieties, if τ is generated by a linearly independent set, then X_τ is nonsingular.

Theorem 1.7. *Let τ be a face of σ such that D_τ is not contained in any $\mathcal{L}_{\alpha,\beta}$, for all non-comparable pairs of join-meet irreducibles (α, β); in other words τ does not contain any $\tau_{\alpha,\beta}$. Then τ is nonsingular.*

sketch of proof. Let $\theta \in D_\tau \cap I_{\alpha,\beta}$.

$$\begin{array}{lcl}
\theta \ngeq A,\, \theta \ngeq B & \Rightarrow & e_\mu - e_A,\, e_\mu - e_B \notin W(\tau) \\
\theta \geq A,\, \theta \ngeq B & \Rightarrow & e_A - e_C,\, e_\mu - e_B \notin W(\tau) \\
\theta \ngeq A,\, \theta \geq B & \Rightarrow & e_\mu - e_A,\, e_B - e_C \notin W(\tau) \\
\theta \geq A,\, \theta \geq B & \Rightarrow & e_A - e_C,\, e_B - e_C \notin W(\tau)
\end{array}$$

□

Remark 1.8. The conjecture holds true for any lattice $\mathcal{L}$ such that $J(\mathcal{L})$ is a grid lattice.

2. Minuscule Lattices

Let G be semi-simple, simply connected algebraic group. Let P be a maximal parabolic subgroup, then P is associated to some fundamental weight.

Definition 2.1. A fundamental weight ω is said to be *minuscule* if $\langle \omega, \beta \rangle \leq 1$ for all $\beta \in R^+$; the maximal parabolic subgroup associated to ω is a *minuscule parabolic subgroup.*

Remark 2.2 (cf [?])**.** Let P be a maximal parabolic subgroup; if P is minuscule then W/W_P is a distributive lattice.

Definition 2.3. For P a minuscule parabolic subgroup, we call $\mathcal{L} = W/W_P$ a *minuscule lattice* and $X(\mathcal{L})$ a *Bruhat-Hibi toric variety*, or a B-H toric variety for short.

Type $\mathbf{A_n}$: Every fundamental weight is minuscule
Type $\mathbf{B_n}$: ω_n
Type $\mathbf{C_n}$: ω_1
Type $\mathbf{D_n}$: $\omega_1, \omega_{n-1}, \omega_n$
Type $\mathbf{E_6}$: ω_1, ω_6
Type $\mathbf{E_7}$: ω_7.

Corollary 2.4. *If $\mathcal{L}$ is a minuscule lattice, then $J(\mathcal{L})$ is a grid lattice.*

Thus, for $\mathcal{L}$ any minuscule lattice, let

$$\Phi = \{(\alpha, \beta) \mid \alpha, \beta \text{ non-comparable join-meet irreducibles in } \mathcal{L}\}.$$

We have completed the proof of the following conjecture from [?].

Theorem 2.5. *For the B-H toric variety $X(\mathcal{L})$,*

$$Sing\, X(\mathcal{L}) = \bigcup_{(\alpha,\beta)\in\Phi} \bar{O}_{\tau_{\alpha,\beta}}.$$

In other words, $X(\mathcal{L})$ is smooth at P_τ (τ being a face of σ) if and only if for each pair $(\alpha, \beta) \in \Phi$, there exists at least one $\gamma \in I_{\alpha,\beta}$ such that $P_\tau(\gamma)$ is non-zero.

3. MULTIPLICITY RESULTS FOR $I_{d,n}$

Corollary 3.1. *The multiplicity of $X_{2,n}$ at P_σ is equal to the Catalan number*

$$Cat_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

Theorem 3.2. *If τ (a face of σ) is a "J-block" of $k+1$ $\sigma_{i,1}$'s, then the multiplicity of $X_{2,n}$ at P_τ is equal to*

$$Cat_{k+2} = \frac{1}{k+3}\binom{2k+4}{k+2}.$$

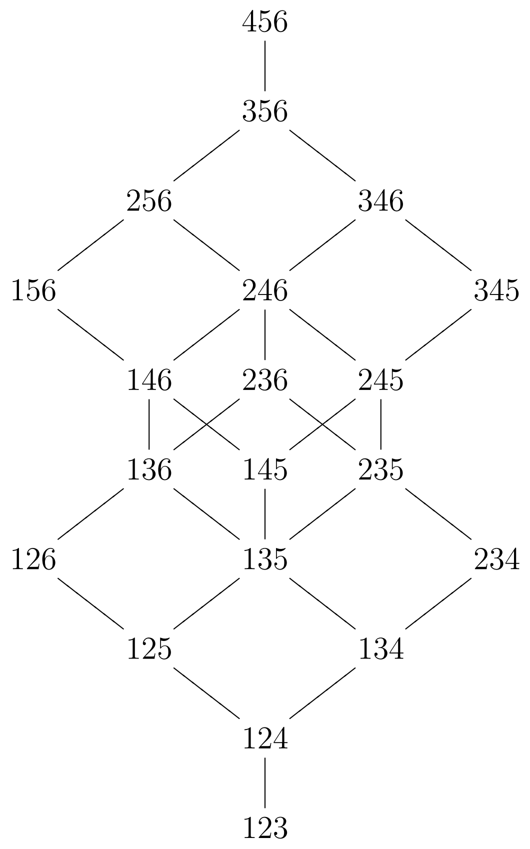
Theorem 3.3. *If $\tau = \tau_1 \cup \tau_2$, where τ_1 and τ_2 are two non-intersecting (and non-consecutive) J-blocks; then*

$$mult_{P_\tau} X_{2,n} = \left(mult_{P_{\tau_1}} X_{2,n}\right) \cdot \left(mult_{P_{\tau_2}} X_{2,n}\right).$$

Theorem 3.4. *The multiplicity of $X_{d,n}$ at P_σ is equal to*

$$\frac{(d\,(n-d))!}{(n-1)(n-2)^2 \dots (n-d)^d (n-d-1)^d \dots (d)^d (d-1)^{d-1} \dots (2)^2 (1)}.$$

Conjecture 1. There exists a multiplicative formula in $X_{d,n}$ for a face that is the union of two disjoint (and non-consecutive) faces.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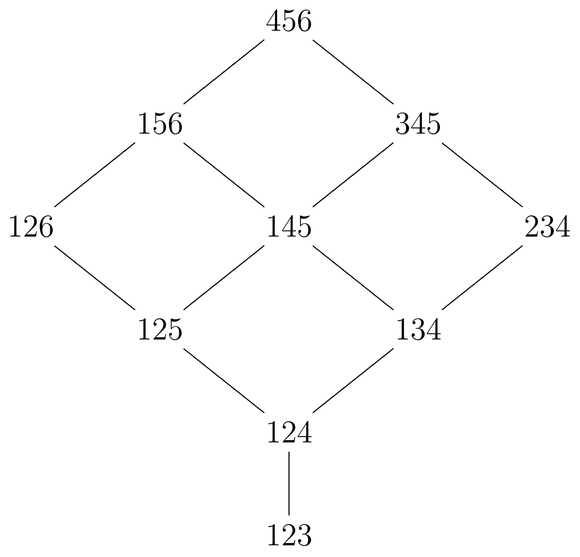
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