# SPHERICAL NILPOTENT ORBITS
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# REPNS. OF SEMISIMPLE LIE GROUPS: AN OVERVIEW

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## Notation

$G$ is a conn., semisimple matrix group; $L(G) = \mathfrak{g}$.
$K \subset G$ is max. compact subgroup; $L(K) = \mathfrak{k}$.
$G_{\mathbf{C}}$ is the complexification of $G$.
$K_{\mathbf{C}}$ is complexification of $K$.

Cartan decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

Complexifications: $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$:

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} \oplus \mathfrak{p}_{\mathbf{C}}$$

$K_{\mathbf{C}}$ acts on $\mathfrak{p}_{\mathbf{C}}$ via the adjoint action (matrix conjugation).

Let $\mathcal{O} = K_{\mathbf{C}} \cdot e$ be a nilpotent orbit in $\mathfrak{p}_{\mathbf{C}}$. By Kostant-Rallis, there is a normal triple $\{x,\ e,\ f\}$, i.e., $x \in \mathfrak{k}_{\mathbf{C}},\ e,\ f \in \mathfrak{p}_{\mathbf{C}}$, with $[x,\ e] = 2e,\ [x,\ f] = -2f,\ [e,\ f] = x$.

**Definition.** $ad(x)(z) := [x,\ z]$.

$\mathfrak{g}_{\mathbf{C}}(j)$ *= the $j$-eigenspace of $ad(x)$. Likewise define* $\mathfrak{p}_{\mathbf{C}}(j)$ *and* $\mathfrak{k}_{\mathbf{C}}(j)$.

**Definition.** *$\mathcal{O}$ has height $j$, if $\mathfrak{g}_{\mathbf{C}}(j) \neq 0$ but $\mathfrak{g}_{\mathbf{C}}(j') = 0$ for all $j' > j$.*

**Definition.** *$\mathcal{O}$ is small if $\mathfrak{p}_{\mathbf{C}}(i) = 0$ for all $i > 2$.*

**Example.** *For $G = SL(n,\ \mathbf{R}),$ or $Sp(n,\ \mathbf{R})$, TFAE: (1) $\mathcal{O}$ has height two; (2) $\mathcal{O}$ is small; (3) $e^2 = 0$; (4) the partition corresponding to $\mathcal{O}$ has all part sizes $\leq 2$.*

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## $K_{\mathbf{C}}$-decompositions of coordinate rings

$\widehat{K_{\mathbf{C}}}$:= set of (equiv. classes of) f.d. irr. reps of $K_{\mathbf{C}}$ (or of $K$).

$\overline{\mathcal{O}}$ is the Zariski closure of $\mathcal{O}$ in $\mathfrak{p}_{\mathbf{C}}$.

$R[\mathcal{O}]$ (resp., $R[\overline{\mathcal{O}}]$) = reg. functions on $\mathcal{O}$ (resp., $\overline{\mathcal{O}}$).

For every nilpotent orbit $\mathcal{O}$, $R[\mathcal{O}]$ and $R[\overline{\mathcal{O}}]$ are completely reducible $K_{\mathbf{C}}$-modules. Thus,

$$R[\overline{\mathcal{O}}] = \sum_{\lambda \in \widehat{K_{\mathbf{C}}}} m_{\overline{\mathcal{O}}}(\lambda) V_\lambda,$$

where each $m_{\overline{\mathcal{O}}}(\lambda)$ is a nonnegative integer (the multiplicity of the irr. repn. $V_\lambda$ in $R[\overline{\mathcal{O}}]$). But there is no general formula for $m_{\overline{\mathcal{O}}}(\lambda)$.

**Remark.** *For the representation theory of $G$, the $K_{\mathbf{C}}$-decomposition of the ring of functions on the normalization of $\overline{\mathcal{O}}$ may be most important.*

## Spherical nilpotent $K_{\mathbf{C}}$-orbits in $\mathfrak{p}_{\mathbf{C}}$

**Definition.** *$\mathcal{O}$ is spherical if a Borel subgroup of $K_{\mathbf{C}}$ has a dense orbit in $\mathcal{O}$*

$$\Longleftrightarrow$$

$\forall\ \lambda \in \widehat{K_{\mathbf{C}}},\ m_{\mathcal{O}}(\lambda) \in \{0,\ 1\}$.

1. (McGovern (1994) and Panyushev (1994)) Classification of spherical orbits for $\mathfrak{g}$ simple and complex.
2. (King, 2003) Classification of spherical orbits for $\mathfrak{g}$ simple and real.

**Remark.** *Examples of small spherical orbits.*

*Orbits of minimal dimension are small and spherical. All height two orbits are small and spherical. Not all small, spherical orbits have height two. Some orbits are small but not spherical.*

## $K$-decomposition and normality of $R[\overline{\mathcal{O}}]$ for $\mathcal{O}$ small and spherical

**Proposition.** *Assume that $\mathcal{O}$ is small and spherical. Let $\mathfrak{n}$ be the nilradical of a Borel subalgebra of $\mathfrak{k}_{\mathbf{C}}$. Then:*

*(1) $\overline{\mathcal{O}}$ is normal.*

*(2) $R[\overline{\mathcal{O}}]^{\mathfrak{n}}$ is a polynomial algebra.*

*(3) There is a subspace $W$ that is $K_{\mathbf{C}}$-conjugate to either $\mathfrak{p}_{\mathbf{C}}(2)$ or $\mathfrak{p}_{\mathbf{C}}(-2)$ such that the generators of $R[\overline{\mathcal{O}}]^{\mathfrak{n}}$ lie in $S(W)$, the symmetric algebra of $W$.*

*Proof.* The proof is based on ideas of Hesselink and Panyushev applied to the standard desingularisation of $\overline{\mathcal{O}}$:

Set $V = V(e) = \sum_{j\geq 2} \mathfrak{p}_{\mathbf{C}}(j)$,
and $\mathfrak{q}_{\mathbf{C}} = \sum\limits_{j\geq 0} \mathfrak{k}_{\mathbf{C}}(j)$.

$\mathbf{Q}_{\mathbf{C}}$ is the conn. subgroup of $K_{\mathbf{C}}$ with Lie algebras $\mathfrak{q}_{\mathbf{C}}$.

The morphism:

$$\pi : K_{\mathbf{C}} \times_{\mathbf{Q}_{\mathbf{C}}} V \to \overline{K_{\mathbf{C}} \cdot e} = K_{\mathbf{C}} \cdot V,$$

defined by $\pi([k, v)] = k \cdot v$ is a desingularization of $\overline{K_{\mathbf{C}} \cdot e}$. The key to proving the proposition is that when $\mathcal{O}$ is small, $V = \mathfrak{p}_{\mathbf{C}}(2)$ and $V$ is a completely reducible $\mathbf{Q}_{\mathbf{C}}$-module. It follows that $R[K_{\mathbf{C}} \times_{\mathbf{Q}_{\mathbf{C}}} V] = R[\overline{K_{\mathbf{C}} \cdot e}]$. □

**Normality of the closure of the principal nilpotent orbit in $\mathfrak{p}_{\mathbf{C}}$ for** $G = SU(2,\ 1)$

$$K_{\mathbf{C}} = GL(2, \mathbf{C}). \quad \mathfrak{p}_{\mathbf{C}} = \left\{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ z & w & 0 \end{pmatrix} |u, v, w, z \in \mathbf{C} \right\}.$$

If $e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathcal{O} = K_{\mathbf{C}} \cdot e$ is the principal orbit. $\mathcal{O}$ is spherical and small but height $= 4$.

$\overline{\mathcal{O}}$= the hypersurface $uz + wv = 0$, the entire nilpotent cone in $\mathfrak{p}_{\mathbf{C}}$.

### Applications to Harish Chandra modules

Assume that $\mathcal{O}$ has height two.

$$\Gamma(\overline{\mathcal{O}}) := \text{the lattice of } K\text{-types in } R[\overline{\mathcal{O}}]$$

**Theorem.** *Assume that $X$ is an irreducible $(\mathfrak{g}_{\mathbf{C}},\ K)$-module and $AV(X)$, the associated variety of $X$, is equal to $\overline{\mathcal{O}}$. Then, the asymptotic directions of the $K$-types of $X$ are given by the cone spanned by $\Gamma(\overline{\mathcal{O}})$ over $\mathbf{R}^+$.*

We illustrate this theorem by considering the $K$-type decomposition of the Speh representation of $SL(4,\ \mathbf{R})$ with Knapp-Vogan parameter $m = -2$, and lowest $K$-type $(1,\ 1)$. We denote this representation by $S[(-2,\ (1,1)]$.

Set $\mathfrak{g} = sl(4,\ \mathbf{R})$ and $\mathfrak{k} = so(4)$. Define the following complex symmetric matrices:

$$Y_1 = \begin{bmatrix} 1 & -i & 0 & 0 \\ -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Y_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \end{bmatrix}$$

$$Y_3 = \begin{bmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ 1 & -i & 0 & 0 \\ -i & -1 & 0 & 0 \end{bmatrix}.$$

Set $\mathcal{O} = K_{\mathbf{C}} \cdot e$ where $e = Y_1 + Y_2$.
$\mathcal{O}$ is a spherical nilpotent orbit of height 2.

Here is a basis for a Borel subalgebra of $so(4,\ \mathbf{C})$:

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$X_{\varepsilon_1-\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix},$$

$$X_{\varepsilon_1+\varepsilon_2} = \begin{pmatrix} 0 & 0 & -1 & i \\ 0 & 0 & i & 1 \\ 1 & -i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}.$$

## The Speh representations

1. In the late 1970's, Birgit Speh proved the existence of an interesting family of irreducible unitary representations of $SL(2n,\ \mathbf{R})$. Their existence had been conjectured by Gelfand and Graev in the 1950's.

2. The Speh representations are important in number theory because they help one decompose $L^2(SL(2n,\ \mathbf{R})/\Gamma)$ for certain congruence subgroups $\Gamma$. It is not easy to write down their explicit realizations on Hilbert spaces.

3. Fact: $AV(S[-2,\ (1,1)]) = \overline{\mathcal{O}}$

## $R[\overline{\mathcal{O}}]^{\mathfrak{n}}$ and $S[-2,\ (1,1)]$

(1) View each $Y_i$ $(i = 1,2,3)$ defined above as a function on $\mathfrak{p}_{\mathbf{C}}$, via $Y_i(Z) = trace(Y_i Z)$.

(2) The subspace $W$ is spanned by $Y_1,\ Y_2$, and $Y_3$

(3) $R[\overline{\mathcal{O}}]^{\mathfrak{n}}$ is generated by $Y_1$ and $Y_1Y_2 - (1/4)Y_3^2$.

(4) $[Y_i,\ Y_j] = Y_iY_j - Y_jY_i = 0$ for all $i,\ j$. So $W$ is a comm. Lie subalgebra of $\mathfrak{p}_{\mathbf{C}}$.

(5) $U(W)$ acts injectively on $S[-2,\ (1,1)]$.

## The $K$-types of $S[(-2,\ (1,1)]$

The $K$-decomposition is:

$$S[(-2,\ (1,1)] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} V(2m+1,\ 2n+1),$$

where $V(2m+1,\ 2n+1)$ is the irr. rep. of $SO(4,\ \mathbf{C})$ with highest weight $(2m+1,\ 2n+1)$.

This decomposition is shown in the diagram below. The dots are integer points in a quadrant (extending to infinity). The dots parametrize f.d. irr. reps of $\mathfrak{k}$. The **circled** dots indicate $K$-types which occur in $S[-2,\ (1,1)]$. The circled dots extend to infinity in the indicated directions. The effect of the generators $Y_1$ and $Y_1Y_2 - (1/4)Y_3^2$ on the $K$-types is also shown.

**The $K$-types of the Speh representation of $SL(4,\ \mathbf{R})$, with Vogan parameter $m = -2$, and lowest $K$ type $(1,\ 1)$**