Formal Group Law

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Cobordism with focus on Formal Group Law (FGL)

1 History Sketch

Started with R. Thom.

Let X, Y, M be oriented manifolds. Then X, Y are corbodent if $\exists M$ such that $\partial M = X \coprod Y$.

$X \sim Y$ iff $\exists M$ such that $\partial M = X \coprod Y$

Ω_n = Abelian group generated by the corbordism classes

addition $[x^n] + [y^n] := [x^n \coprod y^n]$

negation $-[x^n] := [x^n]$ with negation of orientation

$$\Omega = \bigoplus_{n \geq 0} \Omega_n$$

has a ring structure, the multiplication is define by the cartesian product.

Quillen: generalization to maps

Suppose that X is a manifold. We say

$$\begin{array}{ccc} & & Z_1 \\ & & \downarrow{\scriptstyle f_1} \\ Z_0 & \xrightarrow{f_0} & X \end{array}$$

are cordordant if $\exists$

$$\begin{array}{ccc} W & \xrightarrow{b} & X \times \mathbf{R} \\ \uparrow & & \uparrow \\ W_0 & \xrightarrow{b_0} & X \times \{0\} \end{array}$$

such that $b_0 = f_0$ and $b_1 = f_1$.

Identify X with $X \times \{0\}, X \times 1$

Definition.

$$[\underset{f_0=b_0}{W_0 \to X}] = [\underset{f_1=b_1}{W_1 \to X}]$$

Group Structure

Addition: $[Z_1 \xrightarrow{f_1} X] + [Z_2 \xrightarrow{f_2} X] := [Z_1 \coprod Z_2 \xrightarrow{f_1 \coprod f_2} X]$

Negation $-[Z \xrightarrow{f} X] := [Z \xrightarrow{f} X]$ with negation of orientation.

Product $\Omega^*(X) = \oplus_n \Omega^n$

we need a ring structure here

external product $[Z_1 \to X_1] \times [Z_2 \to X_2] := [Z_1 \times Z_2 \to X_1 \times X_2]$

$\Omega^*(X \times X) \xrightarrow{\Delta^*} \Omega^*(X)$

$X \to X \times X$

$X \mapsto (X, X)$

algebraic corbordism (Levine and Morel) $X \in Sch_k$

$\Omega^*(X) =$ abelian group generated by $[Y \to X]$

Line Bundles

$$\begin{array}{c} L_i \\ \downarrow \\ Y \end{array}$$

Key Axiom

[Sect] X is a smooth scheme.

$$\begin{array}{ccc} & \longrightarrow & L_i \\ s \Big| & & \downarrow \\ & \longrightarrow & Y \end{array}$$

$$s^{-1}(0) = Z \to X] = e_1(L) = Z^* Z_*([X])$$

$$Z^* = (\pi^*)^{-1}$$

$$\Omega_*(X) \xrightarrow{\cong} \Omega_{*+1}(L)$$

$LF[s^{-1}(0) \to X] = Z^* Z_*(X)$ cobordant relation
Quillen $[W_0 \to X] = [W_1 \to X]$

Formal Group Law

Let R be a commutative ring with unity: $(F_R, R) = (F, R)$.

$$F_R(u, v) \in R[[u, v]]$$

where $R[[u, v]]$ is the ring of formal power series such that

$$\begin{aligned} F(u,0) = F(0,u) &= u \\ F(u, F(v,w)) &= F(F(u,v), w) \\ F(u,v) &= F(v,u) \end{aligned}$$

We will abbreviate $F(u, v)$ by $u * v$. In this language

$$\begin{aligned} u * 0 = 0 * u &= u \\ u * (v * w) &= (u * v) * w \\ u * v &= v * u \end{aligned}$$

Fact

The exists a universal $(F_{\coprod}, \mathbb{L})$ where

$$\mathbb{L} = \mathbb{Z}[a_{ij} \mid i, j = 0, 1, 2, \ldots]/\sim$$

Universal $\forall (F, R)$ $\exists! \phi : \mathbb{L} \to R$ such that

$$F_{\mathbb{L}} \overset{\phi}{\mapsto} F_R$$

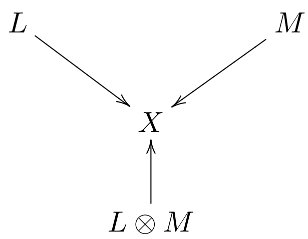
$A : Sch_k \to Ab_*$ (Functor)
$X \mapsto A_*(X)$

$$\begin{array}{l} L \\ \Big\downarrow \longrightarrow \tilde{C}(L) : A_*(X) \to A_{*-1}(X) \\ X \end{array}$$

X has 1_X such that $C_1(L) = \tilde{C}(L)(1_X)$.

First Chrin Class

$C_1(L \otimes M) \overset{\text{how do they relate?}}{\longleftrightarrow} C_1(L) \& C_1(M)$



It turns out that "A" admits a Formal Group Law F_A such that

$$C_1(L \otimes M) = F_A(C_1(L), C_1(W))$$

$$C_1(L \otimes M) \sim C_1(L) \& C_1(M)$$

Examples

3.1 $\Omega_*^+ = \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$
3.2 $\Omega_*^\times = \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$
3.3 $G_0[\beta, \beta^{-1}] := G_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$
3.4 CH
3.5 $CH \otimes \mathbb{Q}[\beta, \beta^{-1}]^{\times\alpha}$

Example 3.1 $\Omega_*^+ = \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$

$(F_{\mathbb{L}}, \mathbb{L})$

$\phi_+ : \mathbb{L} \to \mathbb{Z}$

$\mathbb{Z}[\overline{a_{ij}}]$

$$a_{ij} \mapsto \begin{cases} 1 \text{ if } (i,j) = (0,1) \vee (i,j) = (1,0) \\ 0 \qquad \text{otherwise} \end{cases}$$

ϕ_+ takes $F_{\mathbb{L}} = \sum a_{ij} u^i v^j$ to $F_{\mathbb{Z}}(u,v) = v + u$.

Then

$$\begin{aligned} \Omega_*^+(k) &= \Omega_*(k) \otimes \mathbb{Z} \\ &= \mathbb{L} \underset{\mathbb{L}}{\otimes} \mathbb{Z} \cong \mathbb{Z} \end{aligned}$$

(Fact: $\Omega_*(k) \cong \mathbb{L}$)
(3.1) $F_{\Omega_*^+}(u,v) = u + v$

Example 3.2 $\Omega_*^\times = \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$

$$\phi_\times : \mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$$

$$a_0 \to 1$$

$$a_{10} \to 1$$

$$a_{11} \to -\beta$$

$$a_{ij} \to 0 \qquad \text{otherwise}$$

$\phi_\times$ takes $F_{\mathbb{L}} \to F_{\mathbb{Z}[\beta,\beta^{-1}]}$ where $(u,v) = u + v + a_{11}uv = u + v - \beta uv$ so

$$F_{\Omega_*^\times} = u + v - \beta uv \qquad \text{(multiplicative)}$$

Example 3.3 $G_0[\beta, \beta^{-1}] := G_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$

G_0 is the Grothendieck group.

$$C_1(L \otimes M) \sim C_1(L), C_1(M)$$

$$\begin{array}{c} L \\ \big\downarrow \\ X \end{array} \qquad C_1(L) = (1 - [L^v])\beta^{-1}$$

Then

$$\begin{aligned} C_1(L \otimes M) &= (1 - [(L \otimes M)^v])\beta^{-1} \\ &= (1 - [L^v][M^v])\beta^{-1} \end{aligned}$$

$$\begin{array}{rcl} C_1(L \otimes M) & \overset{?}{\neq} & C_1(L) + C_1(M) \\ & = & (1-[L^v])\beta^{-1} + (1-[M^v])\beta^{-1} \end{array}$$

$$\begin{array}{rcl} C_1(L \otimes M) & = & C_1(L) + C_1(M) - \beta C_1(L) C_1(M) \\ & = & (1-[L^v])\beta^{-1} + (1-[M^v])\beta^{-1} - \beta(1-[L^v])(1-[M^v])\beta^{-1} \\ & = & \beta^{-1}(1-[L^v]+1-[M^v]-(1-[L^v]-[M^v]+[l^v][M^v]) \\ & = & \beta^{-1}(1-[L^v][M^v] \end{array}$$

$$C_1(L \otimes M) = C_1(L) + C_1(M) - \beta C_1(L) C_1(M)$$

$$F_{G(\beta,\beta^{-1})} = u + v - \beta uv \qquad \text{(multiplicative)}$$

Given A to compute $F_A =?$

$$\tilde{C_1} : A_*(X) \to A_{*-1}(X)$$

$$\text{Natural} \quad Q : \tilde{C_1}(L \otimes M) \sim C_1(L), C_1(M)$$

then "A" admits a Formal Group Law F_A sense of

$$F_A(C_1(L), C_1(M)) = C_1(L \otimes M)$$

Example 3.4 $CH_*(X)$

$$\begin{array}{c} L \\ \Big\downarrow \\ X \end{array} \qquad\qquad C_1(L) = D$$

$$C_1(L \otimes M) = C_1(L) + C_1(M)$$

$$L_D \otimes L_{D'} = D + D'$$

$$F_{CH} = u + v$$

$$A = CH \otimes Q[\beta, \beta^{-1}]^{\times\alpha}$$

$$C^A_{\cdot}(L) = \frac{1 - e^{-\beta C_1(L)}}{\beta}$$

$F_{CH^{\times\alpha}_{\beta}} =?$

$$\begin{aligned} C^A_1(L \otimes M) &= \frac{1 - e^{-\beta[C_1\overset{a}{(}L)C_1\overset{b}{(}M)]}}{\beta} \\ &= \frac{1}{\beta}(1 - e^{\beta a}e^{-\beta b}) \end{aligned}$$

so

$$\begin{aligned} \beta C^A_1(L \otimes M) &= (1 - e^{-\beta a} + (1 - e^{-\beta b}) - \beta(1 - e^{-\beta a})(1 - e^{-\beta b}) \\ &= C^A_1(L) + C^A_1(M) - \beta C^A_1(L)C^A_1(M) \end{aligned}$$

$$F_{CH^{(}_{\beta}d)} = u + v - \beta uv$$