Alexander Norm with Applications

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References

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- W. P. Thurston, *A norm for the homology of 3-manifolds*, Mem. Amer. Soc., **59**,1986.

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Notation

- M: A compact, oriented 3-manifold.

- L: A r-component link.

- X: The complement of L.

- $\triangle$: The multivariable Alexander polynomial of X or M.
  $\triangle = \sum_{\alpha \in \text{support}(\triangle)} c_\alpha t^\alpha$

- $N(\triangle)$: The Newton polyhedron of $\triangle$ which is the convex hull of the set of exponents of $\triangle$.

- ϕ: A cohomology class of $H^1(\_, \mathbb{R}) = \mathbb{R}^b$ where $\_$ denotes M or X and b is the first Betti number of M or X.

Alexander and Thurston norms

Definition: (Thurston, 1986) **Thurston norm** In M or X there are compact oriented surfaces. Each cohomology class ϕ determines a set of such surfaces via Poincaré duality. Thurston defines $\chi-$ to be zero if the Euler characteristic χ of the surface is positive or zero and equal to χ otherwise. The absolute value of the minimum Euler characteristic χ of each of the surfaces of the set is called the Thurston norm $\|\phi\|_T$ of ϕ.

$$\|\phi\|_T = |\mathrm{inf}\{\chi-(\Sigma) \mid \Sigma \text{ dual to } \phi\}|$$

The Thurston norm can be extended from the integer values to real values to obtain a Thurston norm unit polyhedron $\mathcal{B}_T$.

$$\mathcal{B}_T = \{\phi \in H^1(\_, \mathbb{R}) \mid \|\phi\|_T \leq 1\}$$

Definition: **Fibered cohomology class**

If $\Sigma \in [\phi]$ is the fiber of a fibration for an integer valued cohomology class, ϕ is called a fibered cohomology class. If ϕ is real valued, it is fibered if all integer valued cohomology classes on the ray from the origin through ϕ are fibered cohomology classes.

Theorem:(Thurston, 1986)**Thurston cone theorem**

The fibered set of cohomology classes is some union of the cones pointed at the origin through the interiors of the top-dimensional faces of the Thurston norm unit ball $\mathcal{B}_T$.

Definition:(McMullen, 2002), **Alexander norm**
Let α and β be the exponents of any two terms of the Alexander polynomial $\triangle$. They are elements of $H_1(X,\mathbb{Z}) = \mathbb{Z}^b$ or $H_1(M,\mathbb{Z})/\text{Tor}(H_1(M,\mathbb{Z})) = \mathbb{Z}^b$ so ϕ acts on them by duality. The Alexander norm $\|\phi\|_A$ is the supremum of $|\phi(\alpha-\beta)|$ over all the exponents of $\triangle$.

$$\|\phi\|_A = \text{sup}_{\alpha,\beta\in\text{support}(\triangle)}\ |\phi(\alpha-\beta)|.$$

The Alexander norm is a special case of a norm on an arbitrary polynomial f called the **f-norm** $\|\phi\|_f$.
New notation: $\|\phi\|_A = \|\phi\|_\triangle$.

- These two norms are actually semi-norms and are closely related.

- The unit ball of both $\mathcal{B}_T$ and $\mathcal{B}_A$ is a polyhedron; a compact, convex set.

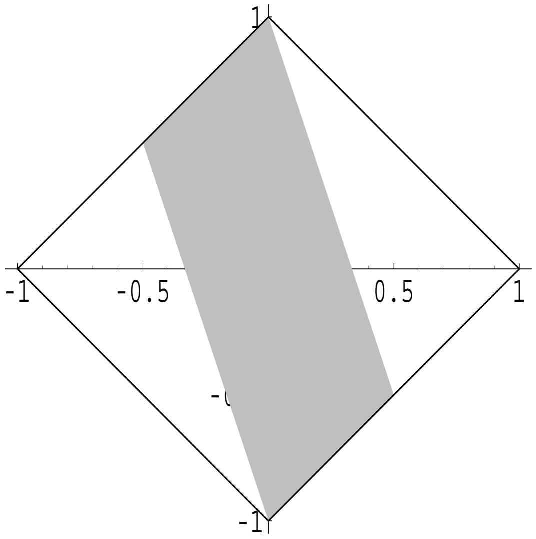
McMullen Conjecture: $\|\phi\|_A = \|\phi\|_T$?

McMullen found that the Alexander and Thurston norms coincided for many calculations.

Theorem:(McMullen, 2002)

- If $b > 1$, then $\|\phi\|_A \geq \|\phi\|_T$.
- If ϕ is a fibered cohomology class, then $\|\phi\|_T = \|\phi\|_A$.

Dunfield found a link such that $\mathcal{B}_T \subset \mathcal{B}_A$, so that $\|\phi\|_T \neq \|\phi\|_A$ for this link.



The Thurston norm unit ball $\mathcal{B}_T$ of Dunfield's link is properly contained in the Alexander norm unit ball $\mathcal{B}_A$. The two Thurston cones of fibered cohomology classes pass through the vertical faces of $\mathcal{B}_T$. The two cones through the horizontal faces form the non-fibered set of cohomology classes.

Definition: **Width function**

The width function of a polyhedron $P \in \mathbb{R}^n$ is the supremum of $|\phi(x-y)|$ where x and y are arbitrary vectors of P.

$$w(P,\phi) = \text{sup}_{\vec{x},\vec{y}\in P}| < \vec{\phi}, \vec{x} - \vec{y} > |.$$

Proposition: The Alexander norm is equal to the width of function of the Newton polyhedron of the Alexander polynomial;

$$\|\phi\|_A = \|\phi\|_\triangle = w(N(\triangle),\phi).$$

This proposition can be proved by using that the supremum takes its values on the vertices of the Newton polyhedron.

Definition:**Minkowski Sum**

If P and Q are two polyhedra, the Minkowski of P and Q, $P+Q$, is the polyhedron made up of all vectors sums $\vec{x}+\vec{y}$ with $\vec{x} \in P$ and $\vec{y} \in Q$.

Two important properties of Minkowski sums of Newton polyhedra $N(f)$ are:

1. $N(f \cdot f') = N(f) + N(f')$.
2. $N(f^n) = nN(f), \forall n > 0$.

Proposition: The width function is a Minkowski linear function on Newton polyhedra so that

1. $w(N(f)+N(f'), \phi) = w(N(f), \phi)+w(N(f'), \phi)$.
2. $w(nN(f), \phi) = n \cdot w(N(f), \phi), \forall n > 0$.

The first property is a standard result from the theory of Minkowski sums of polyhedra and the second is a result of the fact that polyhedron $nN(f)$ is geometrically similar to $N(\triangle)$ with scaling factor n.

This proposition on the width function implies the following corollary for the Alexander norm:

Corollary: The Alexander norm $\|\phi\|_A = \|\phi\|_\triangle$ satisfies the following properties:

1. If $\triangle = f \cdot f'$, then $\|\phi\|_\triangle = \|\phi\|_f + \|\phi\|_{f'}$.
2. If $\triangle = f^n$, then $\|\phi\|_\triangle = n\|\phi\|_f$.

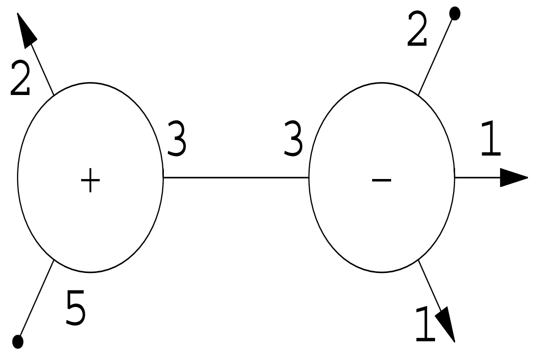
This corollary makes it easy to calculate the Alexander norm as a sum of the Alexander norms of each of the irreducible factors of the Alexander polynomial. Let $\triangle = f_1^{n_1} \dots f_k^{n_k}$, with $n_i > 0, \forall i$, then

$$\|\phi\|_A = \|\phi\|_\triangle = \sum_{i=1}^k n_i \|\phi\|_{f_i}.$$

Eisenbud-Neumann calculus

An Eisenbud-Neumann graph consists of

- **nodes**: Seifert manifolds embedded in the link exterior.
- **arrowhead vertices**: link components.
- **boundary vertices**: solid tubular neighborhoods of Seifert singular fibers.
- **edges**: edges connect nodes and represent the splicing of two Seifert components along a torus.
- **weights**: Attached to each arrowhead and boundary vertex is an integer weight. Attached to each edge are two integers. The integer weights are the degrees of the singular Seifert fibers and Seifert link components. Attached to each node is a sign + or - representing orientations.



An example of an Eisenbud-Neumann graph link with two nodes, three arrowhead vertices and two boundary vertices.T

Each graph link determines a link in a homology 3-sphere. Assume the graph of link L has

- r arrowhead vertices(link components) labeled $v_i, i = 1, \ldots, r$.
- p nodes labeled $v_i, i = r + 1, \ldots, r + p$.
- q boundary vertices labeled $v_i, i = r + p + 1, \ldots r + p + q$.

Alexander polynomial of L

$\triangle = \prod_{i=r+1}^{r+p+q}(t_1^{l_{1i}} \dots t_r^{l_{ri}} - 1)^{(\delta_i - 2)}$.

- $\delta_i, i = r+1, \dots, r+p+q$, denotes the number of edges attached to the node or boundary vertex v_i.

- $l_{ij}, i = 1, \dots, r$ and $j = r+1, \dots r+p+q$ are linking numbers between the ith arrowhead vertex v_i and the jth vertex v_j which is either a node or a boundary vertex. Each linking number is determined by the graph of the link.

Thurston norm of L

$$\|\phi\|_T = \sum_{i=r+1}^{r+p}(\tilde{\delta}_i - 2)|\phi_1 l_{1i} + \ldots + \phi_r l_{ri}|.$$

- $\tilde{\delta}_i = \delta_i - \sum_{i=1}^{q_i} \frac{1}{\alpha_j^i}$.

- q_i = the number of boundary vertices attached to node v_i.

- α_j^i = the integer weight of the jth boundary vertex attached to the ith node.

New Theorem:(2007)The Alexander and Thurston norms coincide for graph link L.

We prove the new theorem by directly calculating the Alexander norm of link L.

- After some simplifications the Newton polyhedron of link L is $N(\triangle) = \sum_{i=r+1}^{r+p} (\tilde{\delta}_i - 2) N(t_1^{l_{1i}} \dots t_r^{l_{ri}} - 1)$.

- Using the Minkowski linearity of the Alexander norm as a width function we find $\|\phi\|_A = \sum_{r+1}^{r+p} (\tilde{\delta}_i - 2) \|\phi\|_{(t_1^{l_{1i}} \dots t_r^{l_{ri}} - 1)}$.

- Evaluating each term in the sum from the original definition of the Alexander norm we find: $\|\phi\|_{(t_1^{l_{1i}} \dots t_r^{l_{ri}} - 1)} = |\phi_1 l_{1i} + \dots + \phi_r l_{ri}|$.

- After substituting the above result, the Alexander norm of L is the same as its Thurston norm: $\|\phi\|_A = \sum_{i=r+1}^{r+p} (\tilde{\delta}_i - 2) |\phi_1 l_{1i} + \dots + \phi_r l_{ri}|$.

This new theorem can be generalized by combining it with two known theorems.

Theorem:(Ozsváth and Szabó, 2006)
Let L be a link in S^3 with an alternating projection, then the Thurston and Alexander norms coincide.

Remark: The proof of this theorem makes use of Floer homology.

Theorem:(Menasco and Thurston, 1982)
Let L be an alternating link in S^3, then either L is a $(2, n)$ torus link or the link complement has a hyperbolic geometry.

A link whose link complement has a hyperbolic geometry is called a simple link. Not all simple links are alternating.

Definition:**JSJ(Jaco-Shalen-Johannson) Decomposition of a 3-manifold**

Every 3-manifold M can be composed as a union along tori of irreducible pieces which are either Seifert-fibered or simple.

New Theorem:(2007)

Let L be a link in S^3, then the Thurston and Alexander norms coincide if either or the following two conditions are met.

- Every component of its JSJ decomposition is a Seifert link.

- Every component of its JSJ decomposition is both simple and alternating.

Link Fibrations Link L is fibered if there is a fibration map $X \to S^1$ with fiber a compact orientable surface in X.

Theorem(Eisenbud-Neumann, 1985)

Fibration Obstruction Criterion

The cohomology class ϕ is a non-fibered cohomology class if and only if it is contained in one of the p hyperplanes $\mathcal{H}_i, i = 1, \ldots, p$ where $\mathcal{H}_i = \{\phi \in H^1(X, \mathbb{R}) \mid \sum_{j=1}^{r} \phi_j l_{ji} = 0\}$.

Sketch of a proof of the Fibration Obstruction Criterion theorem

- L is fibered if and only if each of its p splice components is fibered.

- The Thurston norm of L can be written in terms of the norms of the components;$\|\phi\|_T = \sum_{i=1}^{p} \|\phi\|_T^i$.

- The Thurston norm of the ith splice component is $\|\phi\|_T^i = (\tilde{\delta}_i - 2)|\sum_{j=1}^{r} \phi_j l_{ji}|$.

- The fibration obstruction condition is $\|\phi\|_T^i = 0$ for at least one of $i = 1, \ldots, p$.

- A non-trivial fiber must have Euler characteristic strictly negative $\Rightarrow \|\phi\|_T^i \neq 0$. Hence if $\|\phi\|_T^i = 0$ for some i, then ϕ is a non-fibered cohomology class as claimed.

Definition: **Fibered-Facet**

There is a set of top-dimensional faces(facets) called the *fibered-facets* of the Thurston norm unit ball such that a cohomology class ϕ is a fibered cohomology class if and only it lies in the cone over the interior of one of these top-dimensional faces.

The Fibration Obstruction Criterion and the Thurston cone theorem ⇒

New Theorem:(2007)
Characterization of the Thurston norm unit polyhedron for graph links.

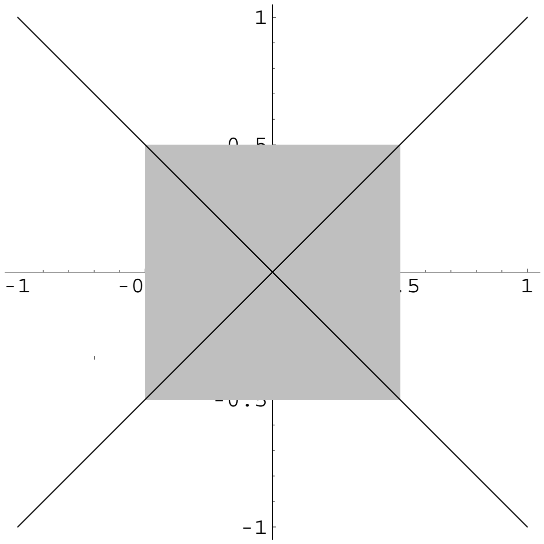
The Thurston norm unit ball of a graph link satisfies the following:

1. Every facet is a fibered facet.
2. Each face of codimension greater than one is a non-fibered face because it is contained in one of the hyperplanes $\mathcal{H}_i, i = 1, \ldots, p$. which make up the non-fibered set of cohomology classes.

The first property of the Thurston norm unit ball can be proved by simple dimensional arguments; the non-fibered sets lie in closed hyperplanes which are of codimension one so they intersect a face in a set of codimension two at most. Hence at least part of each top-dimensional face must have fibered cohomology classes $\Rightarrow$ all the cohomology classes in the interior of each top-dimensional face must be fibered by the Thurston cone theorem.

The cohomology classes in the boundary of the top-dimensional faces must be non-fibered which implies the second property of the Thurston norm unit ball.

Remark: The fibered set of cohomology classes is an open set in the Euclidean metric topology while its complement, the set of non-fibered cohmology classes is closed.



The Thurston norm unit ball of the great circle link with 6 components $L(\mathcal{A})(321456)$ in S^3 and the graph of the two hyperplanes of non-fibered cohomology classes. The complement of these lines form the four Thurston cones of fibered cohomology classes. These lines pass through the cohomology class with all six components equal to one which is the Milnor fibration. Hence there is no Milnor fibration for this link.

The great circle links $L(\mathcal{A})(i_1, \ldots, i_n)$

The great circle $L(\mathcal{A})(i_1, \ldots, i_n)$ can be obtained as follows:

- **Arrangement**: An arrangement of planes is denoted $\mathcal{A}$.
$$\mathcal{A} = \{H_1, \ldots, H_n\}.$$
- **Configuration of an arrangement**: An arrangement $\mathcal{A}$ of planes in in $\mathbb{R}^4 \Rightarrow$ an arrangement of n-skew lines in $\mathbb{R}^3$.
- **Horizontal configuration** $\mathcal{A}(i_1, \ldots, i_n)$: If the n skew lines can be stacked one over the other in planes the configuration is horizontal. The lines can be projected onto a horizontal plane. The order of their slopes in this plane and their order in the vertical stack of planes $\Rightarrow$ a permutation $(i_1, \ldots, i_n)$. The isotopy class is denoted $\mathcal{A}(i_1, \ldots, i_n)$.
- **The link of the configuration** $L(\mathcal{A})(i_1, \ldots, i_n)$: The intersection of the n-skew lines of $\mathcal{A}(i_1, \ldots, i_n)$ with the 3-sphere S^3 make up the link $L(\mathcal{A})(i_1, \ldots, i_n)$.