Riemann-Roch Theorems

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Algebraic Geometry

Classic Algebraic Geometry

k denotes an algebraically closed field

$k^n = \mathbb{A}^n$ denotes n-affine space

For $I \subseteq k[x_1, \ldots, x_n]$ a prime ideal define

$$V(I) = \{\overline{a} \in \mathbb{A}^n \mid f(\overline{a}) = 0 \quad \forall_{f \in I}\}$$

Examples

0. $\mathbb{A}^n = V(0)$
1. $y^2 = x^2 + 3$
2. $y^2 = x^2 + x^3$

When $n = 2$ $V(I)$ is a curve.

If $J \subseteq I$ then $V(I) \subseteq V(J)$.

$$V(I) \longleftrightarrow k[x_1, \ldots, x_n]/I$$

$$\{\text{Affine varieties}\} \longleftrightarrow \{\text{finitely generated nilpotent free rings}\}$$

$$\{\text{Affine Schemes}\} \longleftrightarrow \{\text{Commutative Rings}\}$$

$\mathfrak{a} \rightarrow ring^{\text{Atiyah}}$

$$\text{Spec}(\mathfrak{a}) := \{\mathfrak{p} | \mathfrak{p} \subseteq \mathfrak{a}, \text{ where } \mathfrak{p} \text{ is prime}\}$$

Let $S \subseteq \mathfrak{a}$

$$V(S) = \{\mathfrak{p} \in \mathrm{Spec}(\mathfrak{a}) \ | S \subseteq \mathfrak{p}\}$$

This induces a topology on the ring $\mathfrak{a}$ which is called the Zariski Topology.

Example Let $\mathfrak{a} = k$. Then $\mathrm{Spec}(k) = \{*\}$.

Example Let $\mathfrak{a} = \mathbb{Z}$, then for $\mathfrak{p} \neq 0$ we have $V(\mathfrak{p}) = \{\mathfrak{p}\}$. However $V(0) = \mathrm{Spec}(\mathbb{Z})$. In this case 0 is called a generic point.

$\mathfrak{a}[x_1, \ldots, x_n] = S$

$\mathrm{Spec}(S) = \mathbb{A}^n$

$(X, \mathfrak{a})$

$(X = \mathrm{Spec}(\mathfrak{a}), \theta_x)$ where θ_x is a sheaf.

$\mathbb{P}^n$ projective space, gluing affine space

$\mathbb{R}^1 = \mathrm{Spec}(k[x_2/x_1]) \cup \mathrm{Spec}(k[x_1/x_2])$

(X_1, θ_x)

$U \subseteq X$ is a subvariety

$U \subseteq X$ open, $\theta_x(U) \supseteq U_{1_U} = V(\mathfrak{p})$

$$\mathfrak{p} \subseteq \theta_x(U) \text{ is prime}$$

$$X \vdash - \succ H(X)$$

We associate to a geometric object X an algebraic object $H(X)$.

Two very important objects attached to X are

1 Vector Bundles (K-Theory)

2 Colletion of Subvarieties (Motive Cohomology)

K-Theory

$UB(X) = \{$isomorphism classes of vector bundles on X of finite rank$\}$
It has semigroup structure

$h^0(x) = \{[\varepsilon] \mid$ such that

$$0 \to \varepsilon' \to \varepsilon \to \varepsilon'' \to 0$$

$$[\varepsilon] = [\varepsilon'] + [\varepsilon'']$$

$$\}$$

$G_0(X)$

$X \mapsto CH''(X)$
We want $CH''(X)$ to be a graded ring.

Topological Picture

Let X be a projective variety over $\mathbb{C}$.
(•) $H_*(X, \mathbb{Z}) = \otimes_p(X, \mathbb{Z})$
(•) $H^*(X, \mathbb{Z})$
(•) $\exists$ cup products $H^p \otimes H^q \to H^{p+q}$ so H^* is a ring
(•) $\exists$ a cup product making H_* and aH^* module.

$$\forall r : Y \to X \ \exists \text{a ring homomorphism } r^* : H^*(X) \to H^*(Y)$$

$$\forall F : Y \overset{\text{proper}}{\to} X, \ \exists \text{ a group morphism } F_* : H_*(Y) \to H_*(X)$$

X is non-singular, projective and has dimension n

$$H^p(X) \overset{\cong}{\to} H_{2n-p}(X)$$

$\exists$Churn Classes

k is a fixed field
X is a nonsingular projective variety
$\dim(X) = n$
Def $\mathbb{Z} = \underset{p}{\oplus}\mathbb{Z}_p(X)$ where

$Z_p(X)$ is the free abelian group on subvarieties of X of dimension p

$$\sum n_i[v_i],\ \ n_i \in \mathbb{Z},\ \ v_i \subseteq X(\text{ subvariety})$$

$\mathbb{Z}^*(X) = \oplus\mathbb{Z}^q(X)$ where $\mathbb{Z}^q(X) = \mathbb{Z}_{n-q}(X)$.

$X \mapsto CH'(X) = H_M\ ^0$

$Z^*_{rat} = \{D \in Z^*(X) \mid D = \sum \text{ord } r_i[v_i]\}$

$w \subseteq X$ subvariety where dim $(W) = q+1$

$R(w) = (\theta_w(w))_{group} =$ Field
$F = \frac{a}{b}$ where a, b look like polynomials

$m_i = \dim(\frac{\theta_X w}{a}) - \dim(\frac{\theta_X w}{b})$
$\text{div}(F) \in Z_q(X)$

$$CH^*(X) = \oplus CH^q(X)$$
$$CH^q(X) = \frac{Z^q(X)}{Z^q_{rat}(X)}$$

Examples

1 $Z^*(\text{Spec}(k)) = \mathbb{Z}$.
2 X dim $X = n$, $\mathbb{Z}^*_n(X) = \mathbb{Z}$.

$$Z'(X) = \text{ divisors of } X$$

Let $C \subseteq \mathbb{P}^2$ be a curve.

Class $C =$ number of tangent lives to C through a general point

Theorem (Bezart) Given two plane curves in $\mathbb{P}^2$ given by polynomials F and G, the number of points of $V(F) \cap V(G)$ is at most $\deg(F)\deg(G)$.

$$\#V(F) \cap V(G) \leq \deg(F)\deg(G)$$

$C = V(F) \qquad F(x, y, z) \qquad Q \in \mathbb{P}^2$ where $Q = (a : b : c)$.

$F_{\mathbb{Q}} = aF_x + bF_y + cF_z$

$$\deg F_Q = n - 1 \qquad\qquad C_Q = V(F_Q)$$

$C \cap C_Q = \{p \in C \mid \text{the tangent line of } C \text{ at } P \text{passes through the point } Q\} = n(n-1)$

$C \subseteq \mathbb{P}^2$ non-singular

$L \subseteq \mathbb{P}^2$ a line

$|LnC| = \deg(C)$ gives divisor D_L

So we have a family of divisor paramets by all lines in $\mathbb{P}^2$

$CH^*(X)$ is an intersection Ring on Projective non-singular varieties $W \subseteq$ subvariety.

1 $X \mapsto CH_*(X)$ ab grp
2 We have $X \times Y$ in the intersection ring for any $F : Y \to X$.
3 We have a pairing $CH^p(X) \otimes H^p(X) \to CH^{p+q}(X)$ such that

$$\bullet \forall F : Y \overset{prop}{\to} X \ \exists F_* : CH(Y) \to CH(X) \text{ taking } Z \mapsto F(Z)$$

$$\bullet F_*(Z) = \begin{cases} 0 \text{ if } \dim(Z) = \dim(F(Z)) \\ m_i[F(Z)] \qquad \text{otherwise} \end{cases}$$

<u>Definition</u> $F : Y \to X$

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow & \downarrow{\scriptstyle \gamma F} & \searrow & \\ & & Y \times X & & \\ & \swarrow{\scriptstyle p} & & \searrow{\scriptstyle p_2} & \\ Y & & & & X \end{array}$$

$\gamma(Y) = \Gamma_F =$ graph of$F$

$$\forall F : Y \to X$$

$$F^* : CH^*(X) \to CH^*(Y), \quad Z \mapsto p_{i_*}(\Gamma \cdot p^{-1}(Z)$$

<u>Extra Properties</u>

- Reduction to the diagonal as sets

$$Z \cap Y = (Z \times Y) \cap \Delta$$
$$Z \cdot Y = \Delta^*(Z \times Y)$$

- $P : X \times \mathbb{A}^m \to X$

$p^*$is an isomorphism

$CH_*(\mathbb{A}^m) = CH_0(\mathbb{A}^m) \oplus \ldots \oplus CH_m(\mathbb{A}^m)$

$$\begin{array}{c} \mathbb{A}^m \\ \downarrow \\ * \end{array}$$

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$$Y \xhookrightarrow{\;i\;} X \qquad \text{subvariety}$$

$$u = X - Y \xhookrightarrow{\;j\;} X \qquad \text{open inclusion}$$

$$\begin{array}{ccccc} CH_p(Y) & \xrightarrow{i_*} & CH_p(X) & \xrightarrow{j^*} & CH_r \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

Ex

$$\begin{array}{ccc} Y' & \overset{j}{\hookrightarrow} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$$

We have an exact sequence

$$(X' - Y') \cong (X - Y)$$

$$CH(Y') \to CH_p(X') \oplus CH_p(Y) \to CH_p(X) \to 0$$

$$\begin{array}{c} \varepsilon_x \\ \downarrow \\ [v] \end{array} \qquad \begin{array}{ccc} p^*\varepsilon & \longrightarrow & \varepsilon \\ \downarrow & & \downarrow \\ R(\varepsilon) & \underset{p}{\longrightarrow} & X \end{array} \qquad \begin{array}{c} \varepsilon_x \\ \downarrow \\ X \end{array}$$

vector bundle of rank r

$$p^{-1}(X) = [V]$$
$$p^{-1}(X) = \mathbb{P}(\varepsilon_x)$$
$$L_\varepsilon^v = \text{ line bundle with fiber } \varepsilon_x$$

$\xi\delta = S^*S_*1_{\mathbb{P}(\varepsilon)} \in CH'((P(\varepsilon))$

Proposition $CH(\mathbb{P}(\varepsilon))$ is a $CH^*(X)$ free-module with basis $1, \delta, \ldots, \delta^{r-1}$.