Why care about constructible complexes of sheaves and characteristic cycles?

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Fix a base ring R which is usually $\mathbb{Z}$, $\mathbb{Q}$ is a field is needed. Sometimes it is $\mathbb{R}$ or $\mathbb{C}$.

$$H^k(S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Degree:0

$$\xrightarrow{\partial} \overset{-1}{0} \xrightarrow{\partial} \overset{0}{\mathbb{Z}} \xrightarrow{\partial} \overset{1}{0} \xrightarrow{\partial} \cdots$$

$R_{\pi_*}\underline{\mathbb{Z}}^{\cdot}_x$ is a complex of sheaves on Y.

$$H^k(R(_*\pi)\mathbb{Z}^{\cdot}_x)_y \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{else} \end{cases}$$

$R\pi_*\mathbb{Z}^{\cdot}_x$ is not the sheaf $\underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}}$, there is a monodromy; as y travels around S^1, the two copies of $\mathbb{Z}$ interchange.

The monodromy isomorphism is

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z}$$

$$(x, y) \longmapsto (y, x)$$

<u>General Result</u>

$$\mathbb{H}^k(X^\cdot, \mathcal{A}^\cdot) \xrightarrow{\cong} \mathbb{H}^k(Y^\cdot, R\pi_*\underline{\mathcal{A}}^\cdot)$$

$$X \xrightarrow{\quad\pi\quad} Y$$

$$0 \longrightarrow \mathbb{H}^0(Y^\cdot, R\pi_*\underline{\mathbb{Z}}^\cdot) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{H}^1(Y^\cdot, R\pi_*\underline{Z}^\cdot{}_x) \longrightarrow 0$$

We are always talking about the shaves of R modules.

$R\pi_*\underline{\mathbb{Z}^\cdot_x}$ is an example of a local system,(a locally constant sheaf).

A local system on X wuth stalk M (an R-module) is completely determined by a monodromy representation.:

$$\Pi_1(X, X_0) \xrightarrow[\text{hom}]{\rho} \mathrm{Aut}_R(M)$$

Complexes of sheaves of R-modules on X.
$D(X)$ = derived category of bounded complexes of sheaves on X.

In this case bounded means that $\underline{A}^p = 0$ if $|p|$ is large.
Shift the complex $\underline{\mathbb{Z}}^\cdot$.

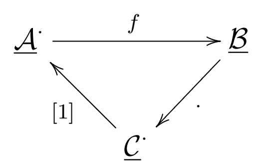
$(\underline{A}[d]^\cdot)^k = \underline{A}^{k+d}$

$\underline{\mathbb{Z}_x}$

$$\overset{-2}{0} \longrightarrow \overset{-1}{0} \longrightarrow \overset{0}{\mathbb{Z}} \longrightarrow \overset{1}{0} \longrightarrow \overset{2}{0}$$

$$(\underline{\mathbb{Z}}^\cdot[1]^{-1}) \qquad \underline{\mathbb{Z}}^0_x \cong \underline{\mathbb{Z}}_x$$

There is a "substitute" for short exact sequences in $D^b(x)$, distinguished triangles:



$$\underline{\mathcal{A}}^{\cdot} \longrightarrow \underline{\mathcal{B}}^{\cdot} \longrightarrow \underline{\mathcal{C}}^{\cdot} \xrightarrow{[1]} \underline{\mathcal{A}}^{\cdot}$$

Devices used for generating long exact sequences (sheaf cohomology).

$$X \xrightarrow{i} Y \qquad\qquad Y - X \overset{j}{\hookrightarrow} Y$$

We have functors

$$i_* \qquad i^* \qquad i_! \qquad i^!$$

$$j_* \qquad j^* \qquad j_! \qquad j^!$$

$$i_! i^! \underline{\mathbb{Z}}^{\cdot}_Y \in D^b(Y)$$

$$\mathbb{H}^k(Y, j_! j^! \underline{\mathbb{Z}}^{\cdot}_Y) \cong H^k(Y, X, \mathbb{Z})$$

$$\begin{array}{ccc} j_! j^! \mathbb{Z}_Y & \longrightarrow & \underline{\mathbb{Z}}^{\cdot}_Y \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & i^* i_* \underline{\mathbb{Z}}^{\cdot}_Y & \end{array}$$

$$\begin{array}{ccccccc} \longrightarrow & \mathbb{H}^k(Y, j_! j^! \mathbb{Z}^{\cdot}_Y) & \longrightarrow & \mathbb{H}^k(Y, \underline{\mathbb{Z}}^{\cdot}_Y) & \longrightarrow & \mathbb{H}^k(Y, i_* i^* \underline{\mathbb{Z}}^{\cdot}) & \longrightarrow \\ & \Big| & & \Big| & & \Big| & \\ & H^k(Y, X, \mathbb{Z}) & & H^k(Y, \mathbb{Z}) & & H^k(X, \mathbb{Z}) & \end{array}$$

Let $\mathcal{U}$ be an open subset of $\mathbb{C}^n$. Look at

$$X \subseteq \mathcal{U}$$

such that

$$X = V(f_1, \ldots, f_k).$$

That is X is the vanishing locus of the complex analytic functions $f_1, \ldots, f_k : \mathcal{U} \to \mathbb{C}$, i.e.

$$X = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0).$$

$$\begin{array}{ccc} V(y-x^2) & \subseteq & \mathbb{C}^2 \\ \downarrow & & \downarrow{\scriptstyle (x,y)} \\ \mathbb{C} & & y \end{array} \qquad \begin{array}{ccc} V(y-x^2)-\{0\} & \subseteq & \mathbb{C}^2 \\ \downarrow & & \downarrow{\scriptstyle (x,y)} \\ \mathbb{C}-\{0\} & & y \end{array}$$

this space has singularities

$$V(y^2 - x^3 - tx^2) \subseteq \mathbb{C}^3$$

places not locally Euclidean

One way to deal with the space is to partition the space into complex manifold pieces (strata).

$$\begin{aligned} \mathcal{S} &= \text{(Whitney) Stratification} \\ &= \left\{V(y^2 - x^3 - tx^2) - V(x,y), V(x,y) - \{\underline{0}\}, \{\underline{0}\}\right\} \end{aligned}$$

the space is "locally trivial" along the strata.
X = complex analytic $\mathcal{S}$= Whitney Stratification of X.

$\underline{\mathcal{A}}^{\cdot} \in D^b_{\mathcal{S}}(X)$ ($\underline{\mathcal{A}}$ is constructible with respect to $\mathcal{S}$) means that $H^k(\underline{\mathcal{A}}^{\cdot})_X$ is a finitely generated R-module and that $\underline{H^k}(\underline{\mathcal{A}}^{\cdot})|_{\mathcal{S}}$ is a local system for all $s \in \mathcal{S}$.

$D^b_C(X)$ = derived category of bounded constructible complexes of sheaves on X.

$\mathbb{H}^k(\underline{\mathcal{A}}^{\cdot})_x \cong \mathbb{H}^k(B^o_\varepsilon(x) \cap X, \underline{\mathcal{A}}^{\cdot})$

$\mathcal{P}^{\cdot} \in D^b_C(X)$ is perverse iff it satisfies the following two conditions:

Support Condition: For all k

$$\dim(\left\{x \in X | H^{-k}(\underline{\mathcal{P}}^{\cdot})_x \neq 0\right\}) \leq k$$

where the set $\left\{x \in X | H^{-k}(\underline{\mathcal{P}}^{\cdot})_x \neq 0\right\}$ is called the support of $\underline{\mathcal{P}}$.

Co-Support Condition For all k

$$\dim(\left\{x \in X | \mathbb{H}^k(B^o_\varepsilon(x) \cap X, B^o_\varepsilon(x) \cap X - x^{\cdot}, \mathcal{P}^{\cdot}) \neq 0\right\} \leq k$$

If X is purely a d-dimensional local complete intersection, then $\underline{\mathbb{Z}}^{\cdot}_x[d]$ is perverse.

Let $f : X \to \mathbb{C}$ be a complex analytic function. The Milner fibre of f at the point $p \in V(f)$, denoted by $F_{f,p}$, is

$$B^o_\varepsilon(p) \cap X \cap f^{-1}(v), \quad \text{where } 0 < |v| << \varepsilon << 1$$

There is a monodromy action on $H^*(F^{\cdot}_{f,p}, \mathbb{Z})$ induced by letting v travel around a circle in $\mathbb{C} - \{0\}$.

$f = y^2 - x^3 - tx^2$

$\psi_f \underline{\mathbb{Z}}^{\cdot}_x =$ sheaf if nearby cycles of $\underline{\mathbb{Z}}^{\cdot}_x$ along f.

$$H^k(\psi_f \underline{\mathbb{Z}}^{\cdot}_x)_p \cong H^k(F^{\cdot}_{f,p}, \mathbb{Z})$$

$\phi_f \underline{\mathbb{Z}}^{\cdot}_x =$ sheaf of vanishing cycles of $\underline{\mathbb{Z}}^{\cdot}_x$ along f.

$$H^k(\phi_f \underline{\mathbb{Z}}^{\cdot}_x) \cong H^{k+1}(B^o_\varepsilon(p) \cap X^{\cdot}, F^{\cdot}_{f,p}, \mathbb{Z}) \cong \tilde{H}^k(F^{\cdot}_{f,p}, \mathbb{Z})$$

We finish with a Theorem:

Theorem If $\mathcal{P}$ is perverse, then $\psi_f \mathcal{P}^{\cdot}[-1]$ ($\psi_f[-1]\underline{\mathcal{P}}$) and $\phi_f \mathcal{P}^{\cdot}[-1]$ ($\phi_f[-1]\underline{\mathcal{P}}$) are perverse.

We need more calculable data that we can associate to a complex of sheaves. Gluing data.

The Morse modules of $S \in \mathcal{S}$ with coefficisents in $\underline{\mathcal{A}}^{\cdot}$ is

$$m_s^k(\underline{\mathcal{A}}^{\cdot}) := \mathbb{H}^k(B_\varepsilon^o(x) \cap N \cap X, B_\varepsilon^{\cdot}(x) \cap N \cap X \cap V(L - L(x) - v), \underline{\mathcal{A}}^{\cdot})$$

where N is a normal slice and $0 < |v| << \varepsilon << 1$.

For each stratum $S \in \mathcal{S}$, consider the closure of the co-normal space to S in $\mathcal{U}$, i.e.

$$\overline{T_S^*\mathcal{U}} = \overline{\{(x,w) \in T^*\mathcal{U} | x \in \mathcal{S} \text{ and } w(T_*S) = 0\}}$$