Examples of Semi-Invariants of Quivers

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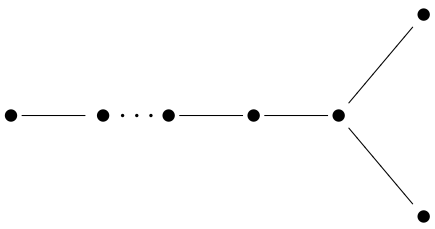
K is an algebraically closed field.

1 Types of Quivers

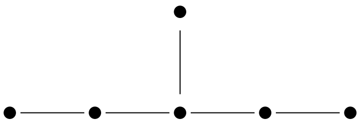
- Quivers with finitely many isomorphism classes of indecomposable representations are of finite representation type. Gabriel's Theorem says that a quiver is of finite representation type if and only if the underlying diagram (the undirected graph obtained by forgetting the direction of arrows) is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.

A_n

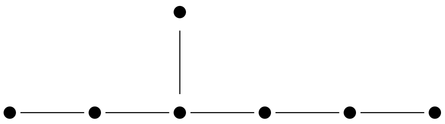
D_n



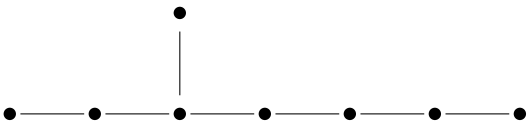
E_6



E_7

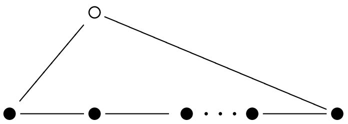


E_8

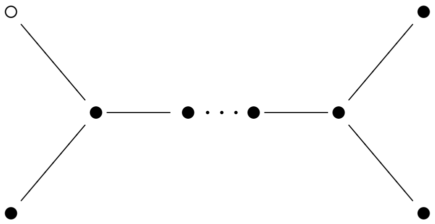


- A quiver is tame if Q is of infinite type but its indecomposable representations occur in each dimension in a finite number of discrete and one-parameter families.
- Theorem: A quiver is of tame type iff the underlying graph is a union of Dynkin graphs and extended Dynkin graphs of type $\widehat{A}, \widehat{D}$ or $\widehat{E}$ as shown below:

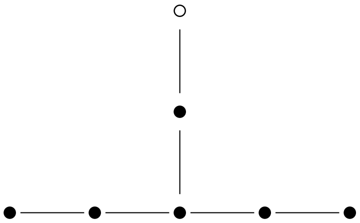
$\widehat{A_n}$



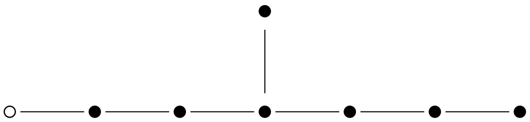
$\widehat{D_n}$



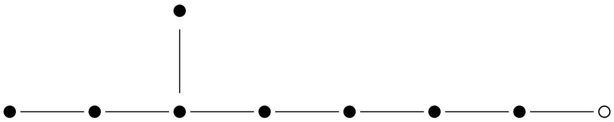
$\widehat{E_6}$



$\widehat{E_7}$



$\widehat{E_8}$



- Every quiver is of either finite type, tame or wild.

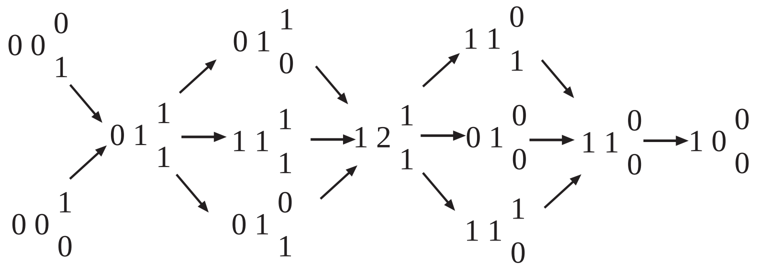
2 Auslander-Reiten Quiver

Definition 2.1 *Definition For a Dynkin or extended Dynkin quiver Q, the Auslander-Reiten quiver AR(Q) is defined as follows: a) the vertices of AR(Q) correspond to the isomorphism classes of indecomposable representations of Q b) the edges of AR(Q) correspond to the irreducible morphisms.*

Example 2.2 *For the D_4 quiver:*

$$\begin{tikzcd}
 & & 3 \\
1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\
 & & 4
\end{tikzcd}$$

The Auslander-Reiten quiver is the following (with vertices the dimension vectors of the corresponding representations):



- From the AR quiver we can easily read off the dimensions of the space $Hom_Q(U,V)$ and $Ext_Q(U,V)$ for the indecomposable representations U and V.
- Auslander-Reiten translations are defined to be the compositions of D and Tr (the dual and the transpose). with $\tau^+ = D \circ Tr$ and $\tau^- = Tr \circ D$. In the AR quiver τ^- translates to the right and τ^+ translates to the left.
- Each homomorphism between U and V is a linear combination of paths in AR(Q). The dimension of $Ext_Q(U,V)$ reduces by Auslander-Reiten duality to the dimension of $Hom_Q(V,\tau^+(U))$ for U not projective or $Hom_Q(\tau^- V, U)$ for V not injective. If U is projective or V is injective then $Ext_Q(U,V) = 0$.
- In the Dynkin case, the AR quiver will give all of the indecomposable representations.
- To each quiver, and a field K, we can associate a path algebra KQ, and the category of quiver representations is equivalent to the category of modules over the path algebra.

3 Questions about semi-invariants

Problems to consider:

We consider quivers only of finite type or tame, since it is unrealistic to define semi-invariants for the wild case, even in the hereditary case. An algebra A is hereditary if projective dimension is at most 1. For quivers without relations, associated path algebras are all hereditary.

For quivers without relations:

- When is the ring of semi-invariants a polynomial ring or a hypersurface? (When Q is Dynkin, the ring of semi-invariants are all polynomial rings) (When Q is extended Dynkin then the ring of semi-invariants is either a polynomial ring or a hypersurface)
- When the ring of semi-invariants is a polynomial ring, when is it a complete intersection?
  Definition: If SI(Q,β) is the algebra $K[X_1,\dots,X_n]/I(Q,\beta)$ of polynomial functions on an affine variety V. If dim $V = n - r$ and $I(Q,\beta)$ is generated from r polynomials from $K[X_1,\dots,X_n]$, then SI(Q,β) is called a complete intersection.
- For which other quivers and dimension vectors is the ring of semi-invariants a polynomial ring?

For quivers with relations:

- What is the characterization of algebras of finite representation type or tame in terms of semi-invariants?

4 What I am working on

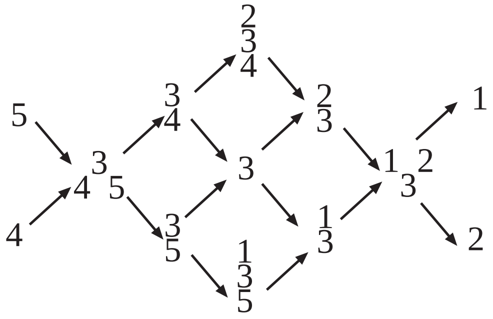
What are the semi-invariants for the tilting and iterated tilting algebra of A_n?

Example 4.1 *Let* Q *be a quiver with the set of vertices* $Q_0 = \{1, 2, 3, 4, 5\}$ *and the set of arrows* $Q_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *with relations* $\alpha_3\alpha_1 = \alpha_4\alpha_2 = 0$

$$\begin{tikzcd}
1 \arrow[dr, "\alpha_1"'] & & 2 \arrow[dl, "\alpha_2"] \\
& 3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_4"] & \\
4 & & 5
\end{tikzcd}$$

Let the dimension vector for Q be $\beta = (1, 1, 2, 1, 1)$. This is an example of a tilted algebra for a non-equioriented A_5 quiver. (For a tilting algebra of an equioriented A_n, it is known that ring of semi-invariants is a polynomial ring.)

The Auslander-Reiten quiver for Q is:



The Euler matrix for the quiver with relations is:

$$E = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The rings of semi-invariants on components of representation spaces are generated by c^V where V is an indecomposable of projective dimension 1 with $< d(V), \beta >= d(V)E\beta^T = 0$. Thus $d_4 = d_5 = 0$

Since we can see all the indecomposable representations in the AR quiver, the semi-invariants are generated by c^V with V indecomposable with

$$d(V) \in \{(0,0,1,0,0),(0,1,1,0,0),(1,1,1,0,0),(1,0,1,0,0)\}$$

.

For (0,0,1,0,0) we have this corresponds to the weight (0,0,1,-1,-1) (given by $< d(V), - >= d(V)E$) similarly we have:

$$(0,1,1,0,0)E = (0,1,0,-1,0)$$
$$(1,1,1,0,0)E = (1,1,-1,0,0)$$
$$(1,0,1,0,0)E = (1,0,0,0,-1)$$

Thus the generators for the ring of semi-invariants are:

$$\delta_1 = det(V(\alpha_3) + V(\alpha_4) : V(3) \to V(4) \oplus V(5))$$

$$\delta_2 = det(V(\alpha_3) \circ V(\alpha_2) : V(2) \to V(4))$$
$$\delta_3 = det(V(\alpha_1) + V(\alpha_2) : V(1) \oplus V(2) \to V(3))$$
$$\delta_4 = det(V(\alpha_4) \circ V(\alpha_1) : V(1) \to V(5))$$

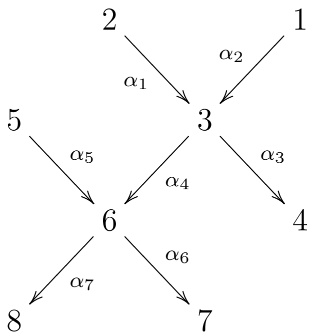
Looking at the linear map

$$V(\alpha_4) \circ V(\alpha_1) \oplus V(\alpha_3) \circ V(\alpha_2) : V(1) \oplus V(2) \to V(4) \oplus V(5)$$

its determinant is $\delta_3\delta_4$ but factoring the morphism through $V(3)$ one can se that its determinant is equal to $\delta_1\delta_2$ as well. Thus the semi-invariants fulfill a relation $\delta_1\delta_2 - \delta_3\delta_4 = 0$. Therefore we can conclude the the the SI(Q,β) is a hypersurface $K[\delta_1,\delta_2,\delta_3,\delta_4]/(\delta_1\delta_2 - \delta_3\delta_4)$.

Notice that in the AR quiver, the generators form a rectangle and the relations can be easily seen. This seems to be the general case with this type of quiver, as seen in the following example:

Example 4.2 *Let* Q *be a quiver with the set of vertices* $Q_0 = \{1,2,3,4,5,6,7,8\}$ *and the set of arrows* $Q_1 = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}$ *with relations* $\alpha_4\alpha_1 = \alpha_3\alpha_2 = \alpha_6\alpha_4 = \alpha_7\alpha_5 = 0$

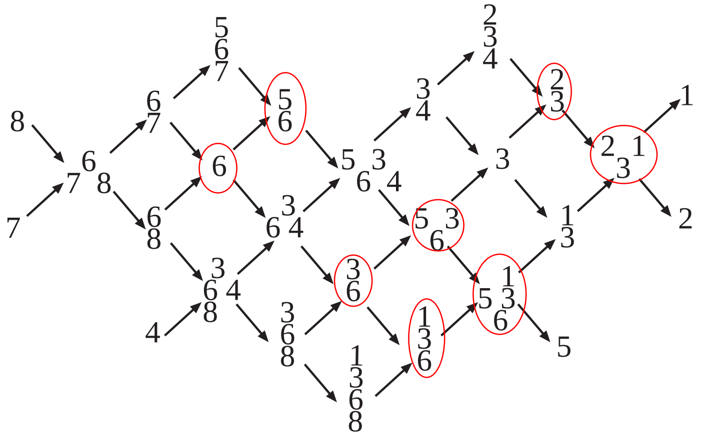


Let the dimension vector for Q be $\beta = (1, 1, 2, 1, 1, 2, 1, 1)$. The Euler matrix for this quiver is

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Again we can determine the c^V's by looking at the AR quiver to see which indecomposables V have dimension vector such that $< d(V), \beta >= 0$ and have projective dimension 1.

The A-R quiver for Q follows, with indecomposables orthogonal to β and projective dimension 1 circled in red.



Calculating weights of these semi-invariants, we see that:

$\delta_1 = det(V(1) \oplus V(2) \to V(3)))$

$\delta_2 = det(V(2) \to V(4))$

$\delta_3 = det(V(5) \to V(7))$

$\delta_4 = det(V(6) \to V(7) \oplus V(8))$

$\delta_5 = det(V(1) \oplus V(5) \to V(6))$

$\delta_6 = det(V(1) \to V(8))$

$\delta_7 = det(V(3) \oplus V(5) \to V(4) \oplus V(6))$

$\delta_8 = det(V(3) \to V(4) \oplus V(8))$

with relations $\delta_4\delta_5 - \delta_3\delta_6 = 0$, $\delta_1\delta_7 - \delta_2\delta_5 = 0$ $\delta_2\delta_6 - \delta_1\delta_8 = 0$, $\delta_5\delta_8 - \delta_6\delta_7 = 0$, $\delta_4\delta_7 - \delta_3\delta_8 = 0$

Thus $\mathrm{SI}(Q, \beta) = K[\delta_1, \ldots, \delta_8]/(\delta_4\delta_5 - \delta_3\delta_6, \delta_1\delta_7 - \delta_2\delta_5, \delta_2\delta_6 - \delta_1\delta_8, \delta_5\delta_8 - \delta_6\delta_7, \delta_4\delta_7 - \delta_3\delta_8)$ Some analysis reveals that this is not a complete intersection.